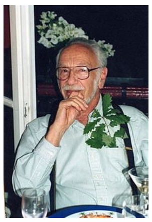
15-251
Great Ideas in Theoretical Computer Science

Lecture 14:
Graphs IV: Stable Matchings

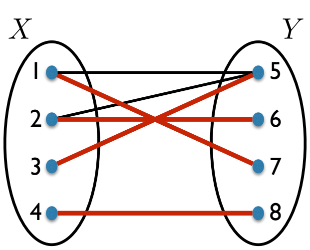


October 12th, 2017

Hall's Theorem

Characterization for perfect matchings

Often we are interested in perfect matchings.



<u>An obstruction:</u>

$$|X| \neq |Y|$$

Characterization for perfect matchings

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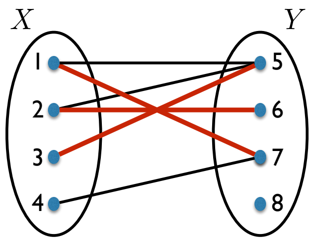


An obstruction:

If $|X| > |Y|$, we cannot "cover" all the nodes in X.

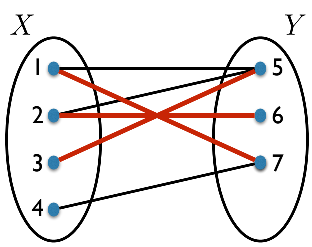
If $|X| > |N(X)|$, we cannot "cover" all the nodes in X.

Characterization for perfect matchings

Often we are interested in perfect matchings.

$S = \{1,3,4\}$

$N(S) = \{5,7\}$



An obstruction:

For $S \subseteq X$:

if $|S| > |N(S)|$, we cannot "cover" all the nodes in S.

Characterization for perfect matchings

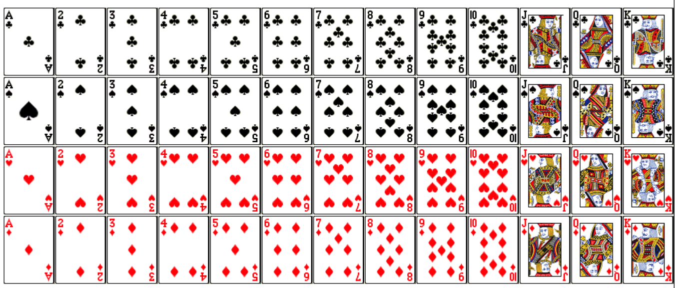
Is this the only type of obstruction?

Theorem [Hall's Theorem]:

Corollary:

An application of Hall's Theorem

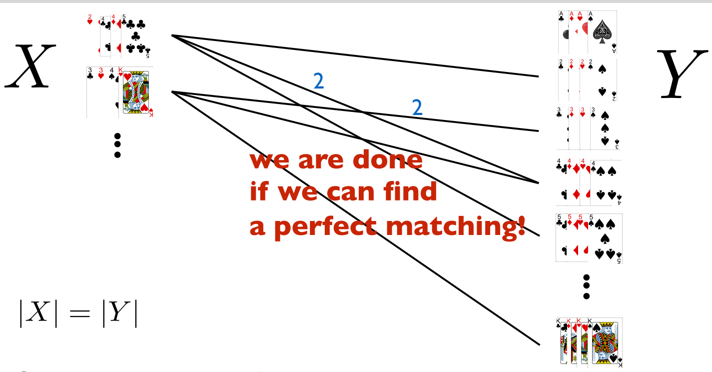
Rank: 1 2 3 4 5 6 7 8 9 10 J Q K



Suppose a deck of cards is dealt into 13 piles of 4 cards each.

Claim: there is always a way to select one card from each pile so that you have one card from each rank.

An application of Hall's Theorem

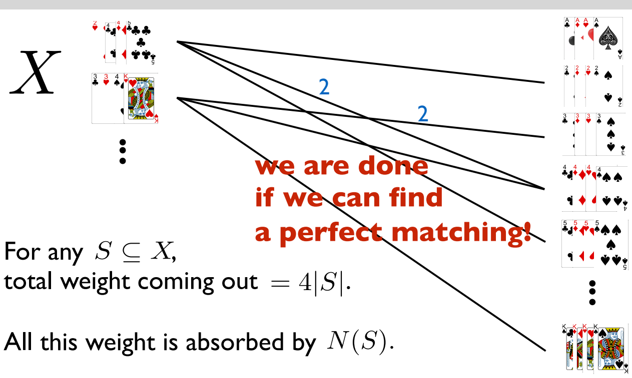


$|X| = |Y|$

So we want to show:

For any $S \subseteq X$, $|S| \leq |N(S)|$.

An application of Hall's Theorem



For any $S \subseteq X$,
total weight coming out $= 4|S|$.

All this weight is absorbed by $N(S)$.

Each $y \in N(S)$ absorbs ≤ 4 units of this weight.

$\implies N(S)$ absorbs ≤ $4|N(S)|$ units. $\implies 4|S| \leq 4|N(S)|$

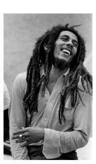
Stable matching problem

2-Sided Markets

A market with 2 distinct groups of participants each with their own preferences.

2-Sided Markets

1. Moogle
2. Macrosoft
3. KLG
4. Umbrella Corporation



Macrosoft

1. Alice
2. Bob
3. Charlie
4. David

Moogle

Umbrella Corporation

"Our business is life itself..."

•
•
•

1. Bob
2. David
3. Alice
4. Charlie

Other examples:

medical residents - hospitals

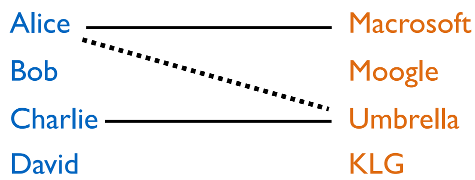
students - colleges

professors - colleges

:

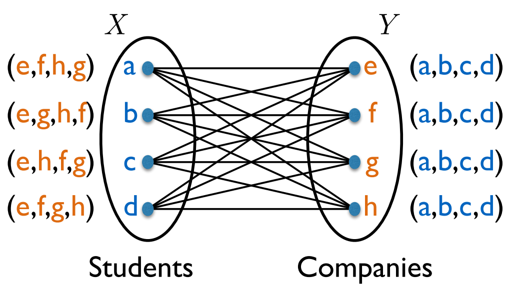
Aspiration: A Good Centeralized System

What can go wrong?



Formalizing the problem

An instance of the problem can be represented as a *complete bipartite graph* + *preference list of each node.*

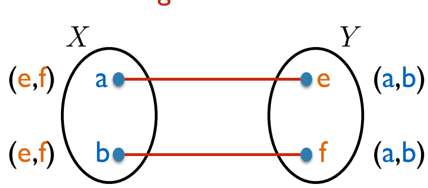


$$|X| = |Y| = n$$

Goal:

Formalizing the problem

What is a stable matching?



A variant: Roommate problem

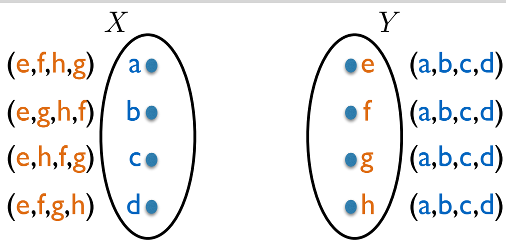
A non-bipartite version

(c,b,d) a • • c (b,a,d)

(a,c,d) b • • d (a,c,b)

Does this have a stable matching?

Stable matching: Is there a trivial algorithm?



Trivial algorithm:

The Gale-Shapley proposal algorithm

While there is a man **m** who is not matched:

- Let **w** be the highest ranked woman in **m**'s list to whom **m** has not proposed yet.
- If **w** is unmatched, or **w** prefers **m** over her current match:
  - Match **m** and **w**. (The previous match of **w** is now unmatched.)

Cool, but does it work correctly?

- Does it always terminate?
- Does it always find a stable matching? (Does a stable matching always exist?)

Gale-Shapley algorithm analysis

Theorem:
The *Gale-Shapley proposal algorithm* always terminates with a stable matching after at most n^2 iterations.

A *constructive* proof that a stable matching always exists.

3 things to show:

Gale-Shapley algorithm analysis

1. Number of iterations is at most n^2.

Gale-Shapley algorithm analysis

2. The algorithm terminates with a perfect matching.

If we don't have a perfect matching:

A man is not matched

$\Longrightarrow$ All women must be matched

$\Longrightarrow$ All men must be matched.

Contradiction

Gale-Shapley algorithm analysis

2. The algorithm terminates with a perfect matching.

If we don't have a perfect matching:

A man is not matched

$\Longrightarrow$ All women must be matched

$\Longrightarrow$ All men must be matched.

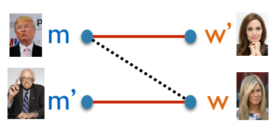
Contradiction

Gale-Shapley algorithm analysis

3. The matching has no unstable pairs.

"Improvement" Lemma:
(i) A man can only go down in his preference list.
(ii) A woman can only go up in her preference list.

Unstable pair:
(m,w) unmatched
but they prefer each other.



□

Further questions

Theorem:
The *Gale-Shapley proposal algorithm* always terminates with a stable matching after at most n^2 iterations.

Does the order of how we pick men matter?
Would it lead to different matchings?

Is the algorithm "fair"?
Does this algorithm favor men or women or neither?

Further questions

m and **w** are *valid partners* if there is a stable matching in which they are matched.

best(**m**) = highest ranked valid partner of **m**

Theorem:

Further questions

worst(**w**) = lowest ranked valid partner of **w**

Theorem:

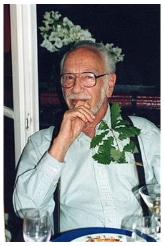
Real-world applications

Variants of the Gale-Shapley algorithm is used for:

- matching medical students and hospitals
- matching students to high schools (e.g. in New York)
- matching students to universities (e.g. in Hungary)
- matching users to servers

⋮

The Gale-Shapley Proposal Algorithm (1962)



Nobel Prize in Economics   2012

"for the theory of stable allocations and the practice of market design."