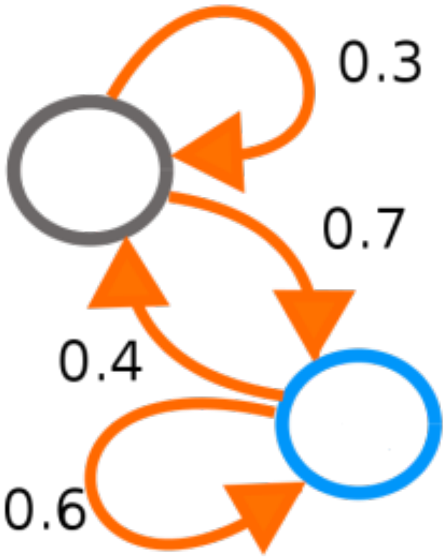
15-251
Great Theoretical Ideas in Computer Science

Lecture 23:
Markov Chains

November 17th, 2015



My typical day (when I was a student)

9:00am

Work

$$f(x) = \sum_{S \subseteq [n]} \widehat{f}(S) \chi_S(x)$$

My typical day (when I was a student)

9:01am

Work

$$f(x) = \sum_{S \subseteq [n]} \widehat{f}(S)\chi_S(x)$$

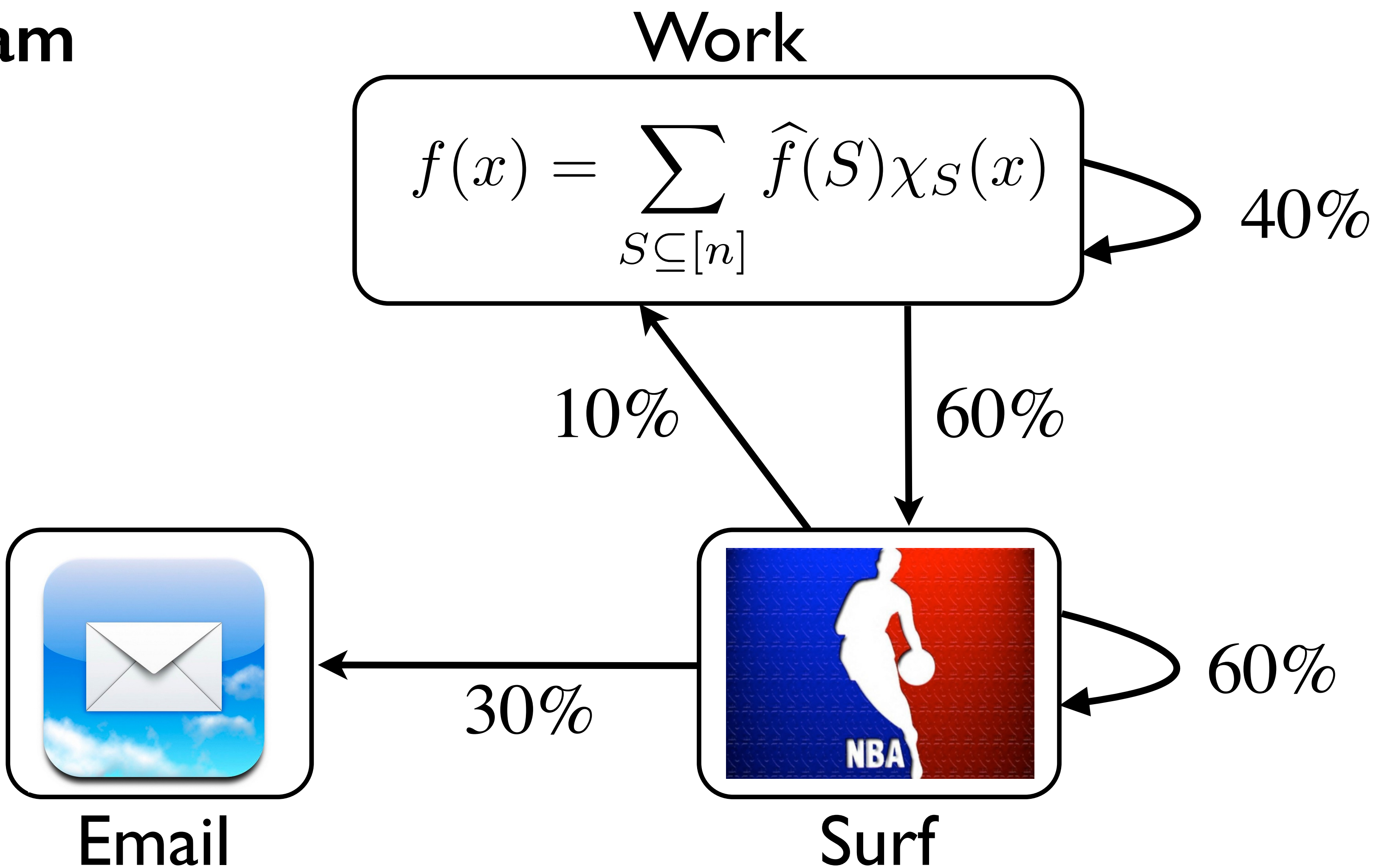
40%

60%

Surf

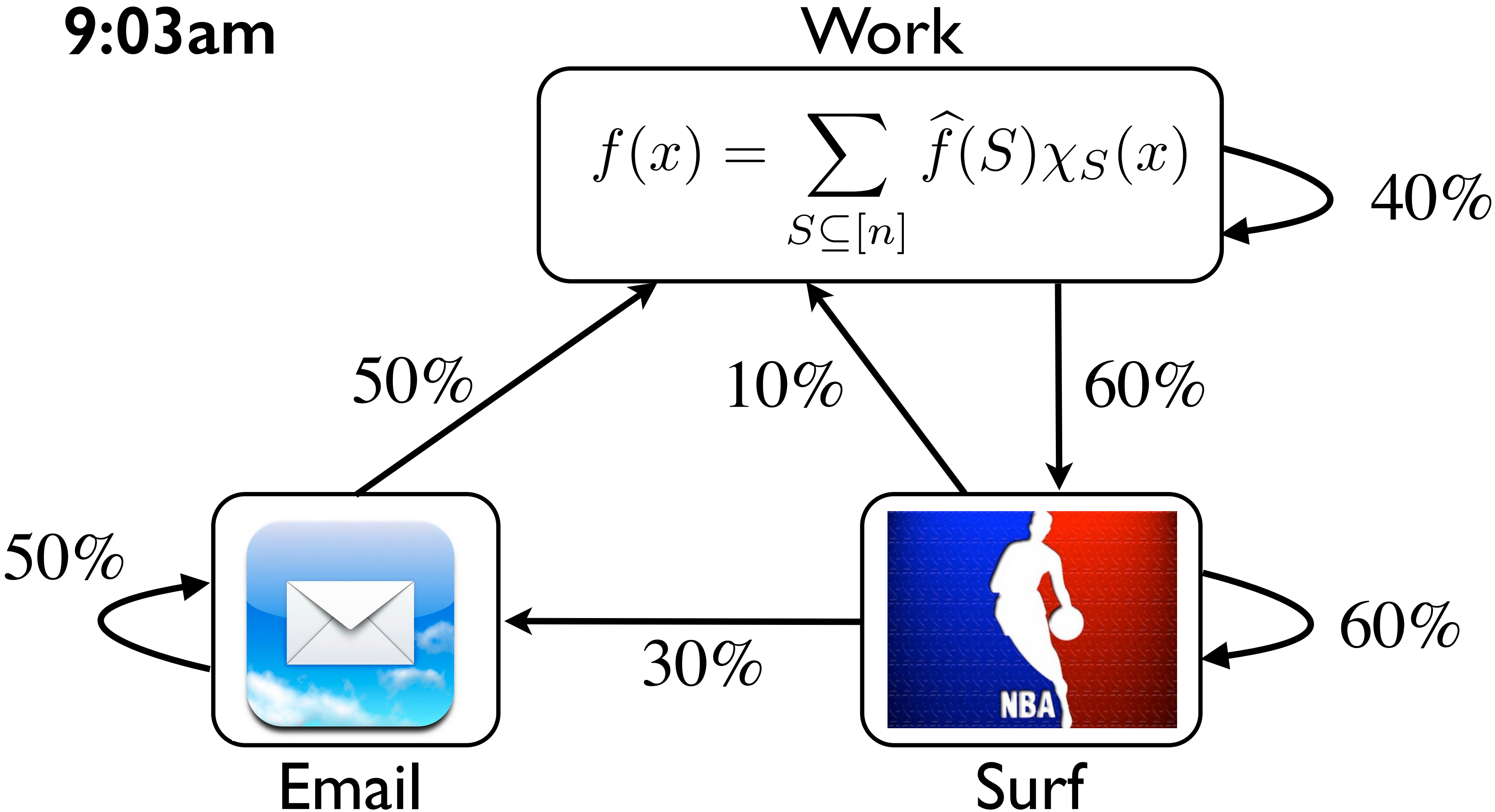
My typical day (when I was a student)

9:02am



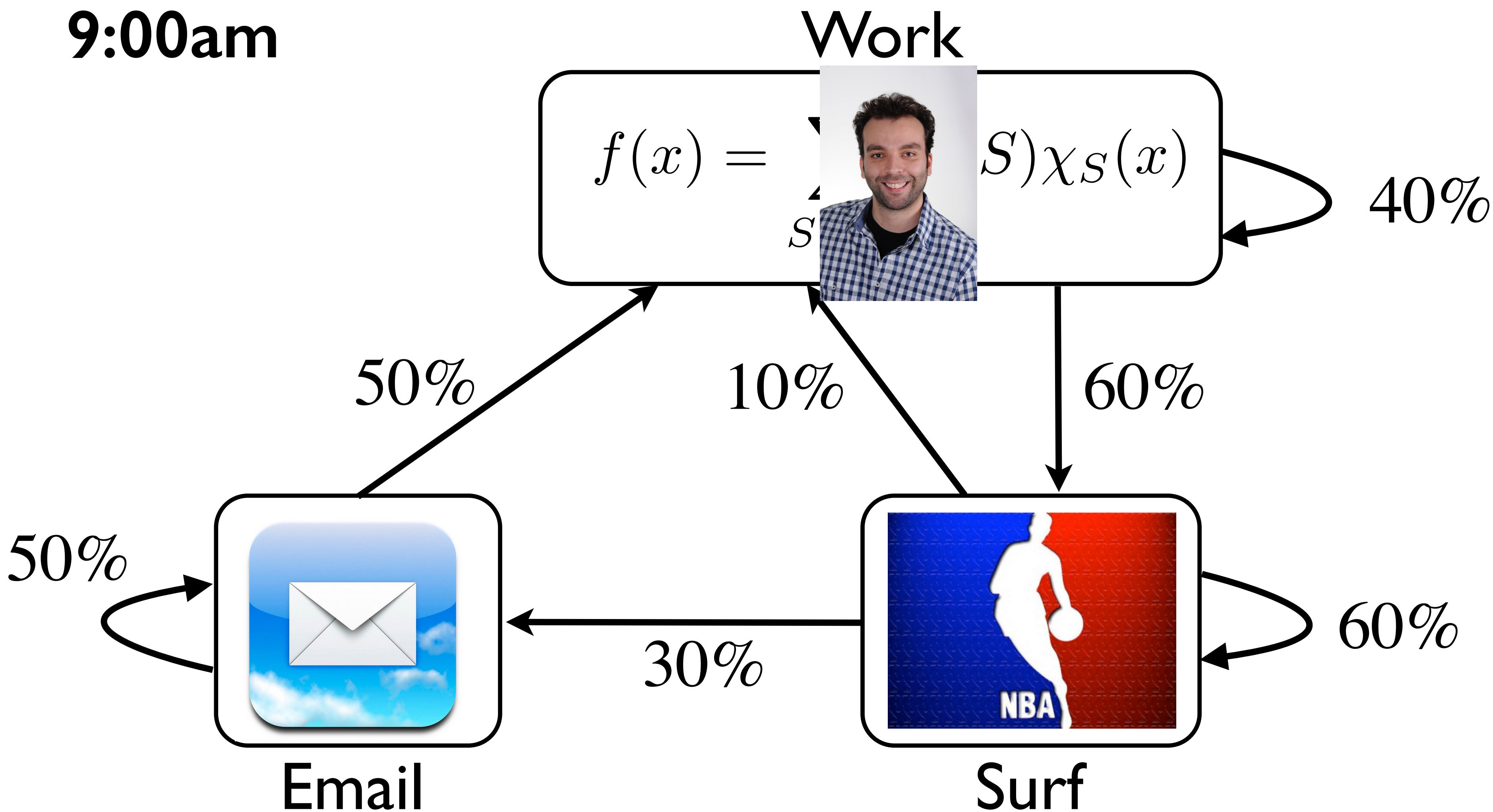
My typical day (when I was a student)

9:03am



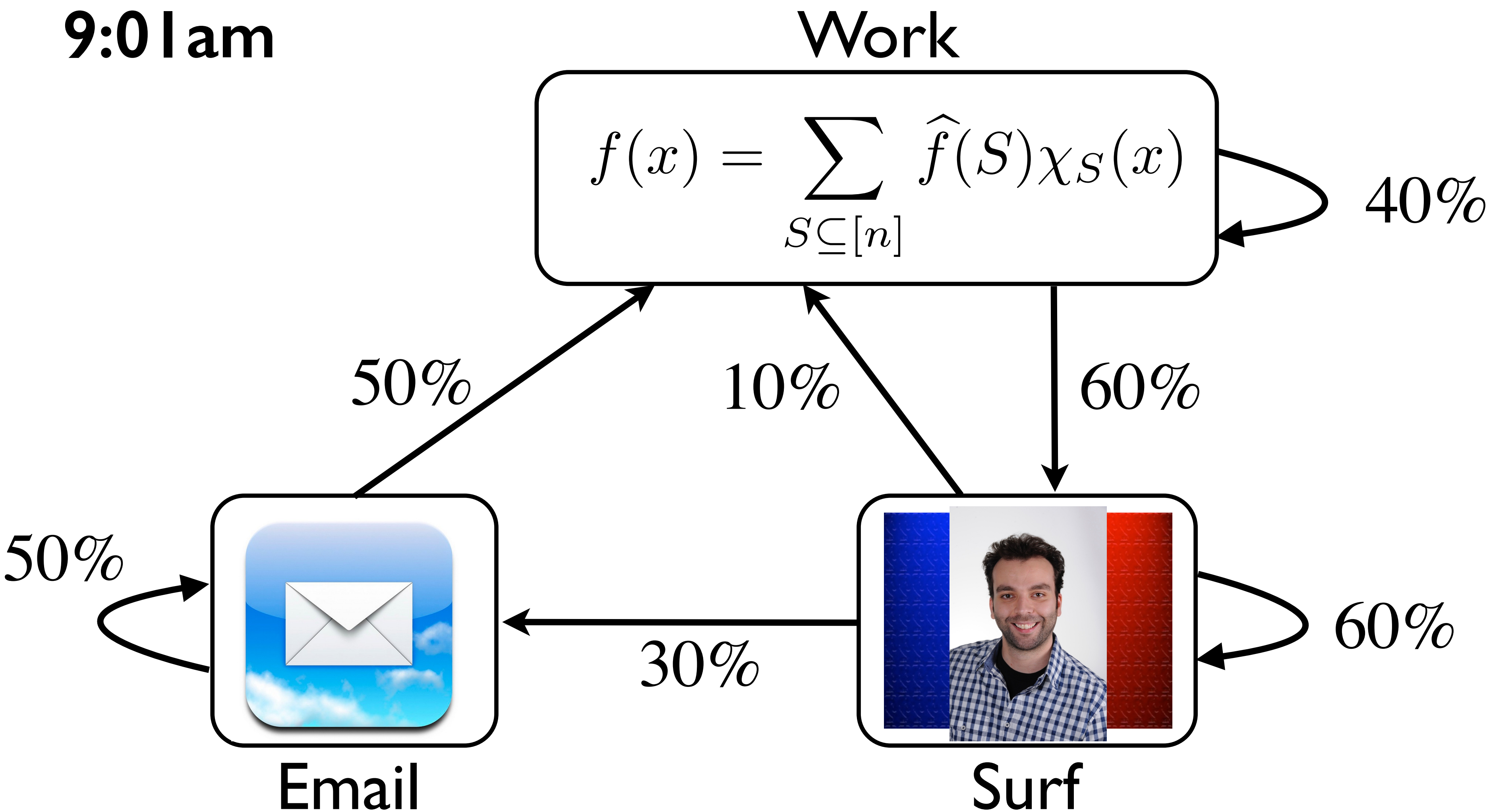
My typical day (when I was a student)

9:00am



My typical day (when I was a student)

9:01am



My typical day (when I was a student)

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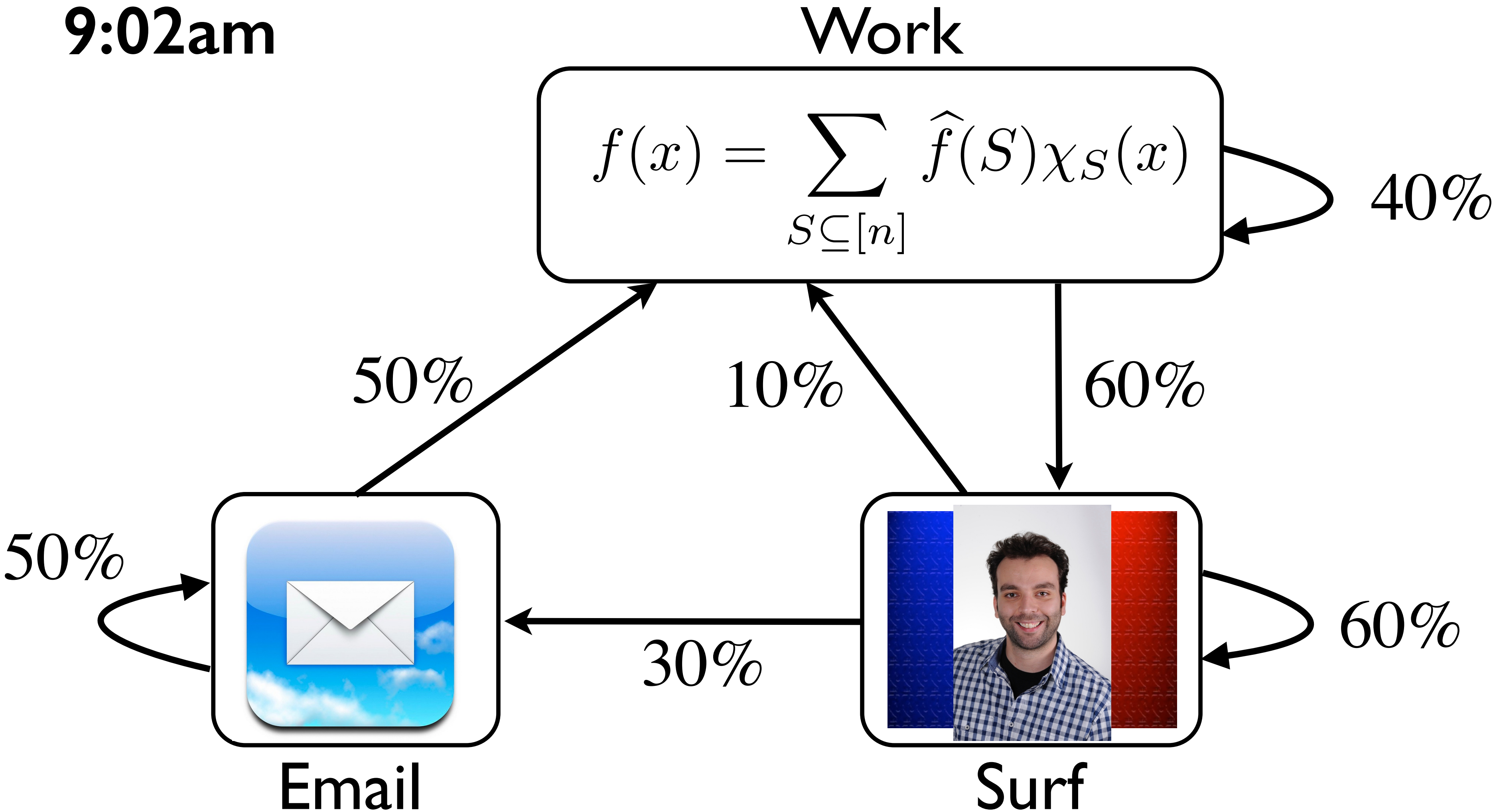
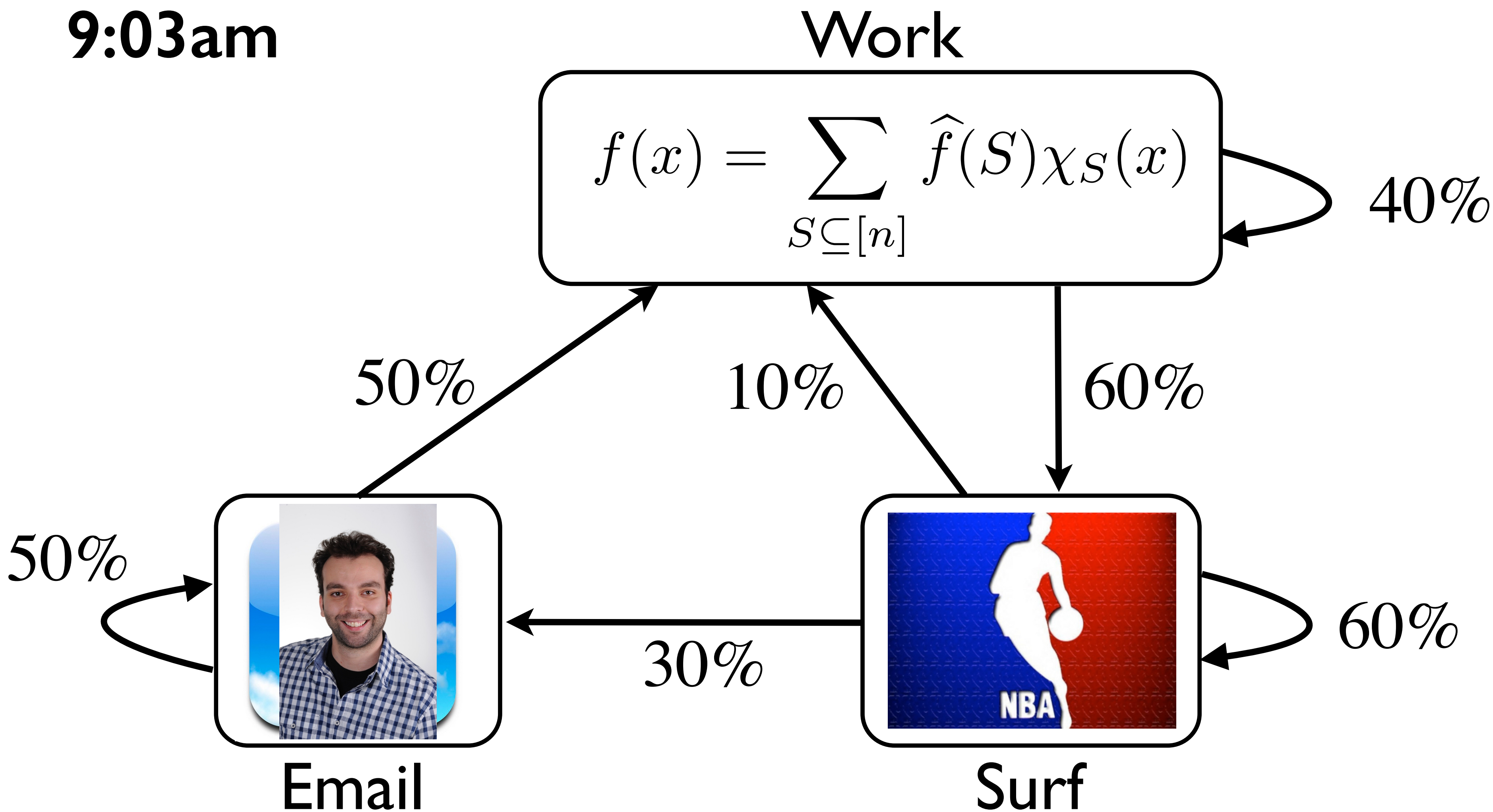


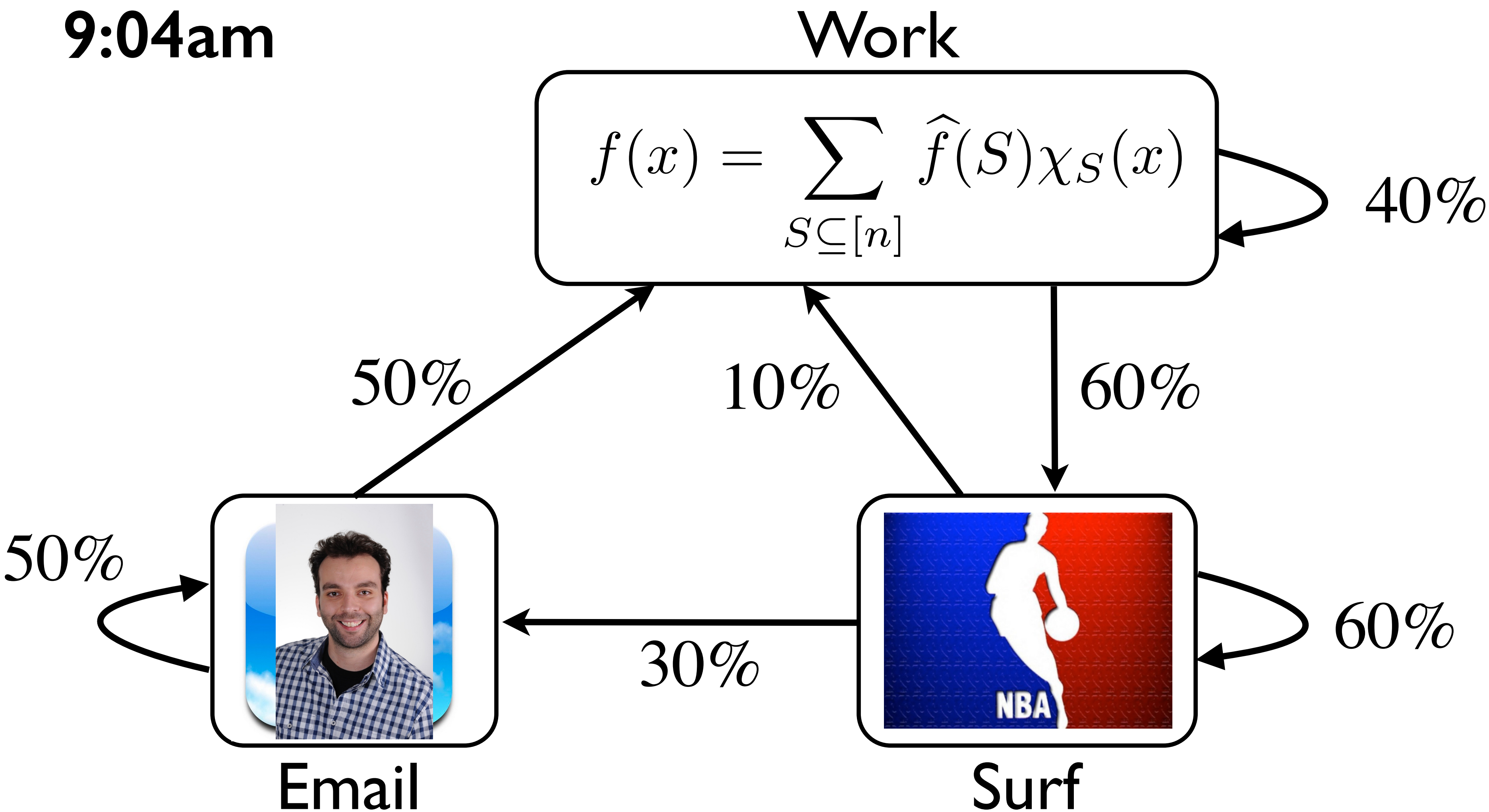
My typical day (when I was a student)

9:03am



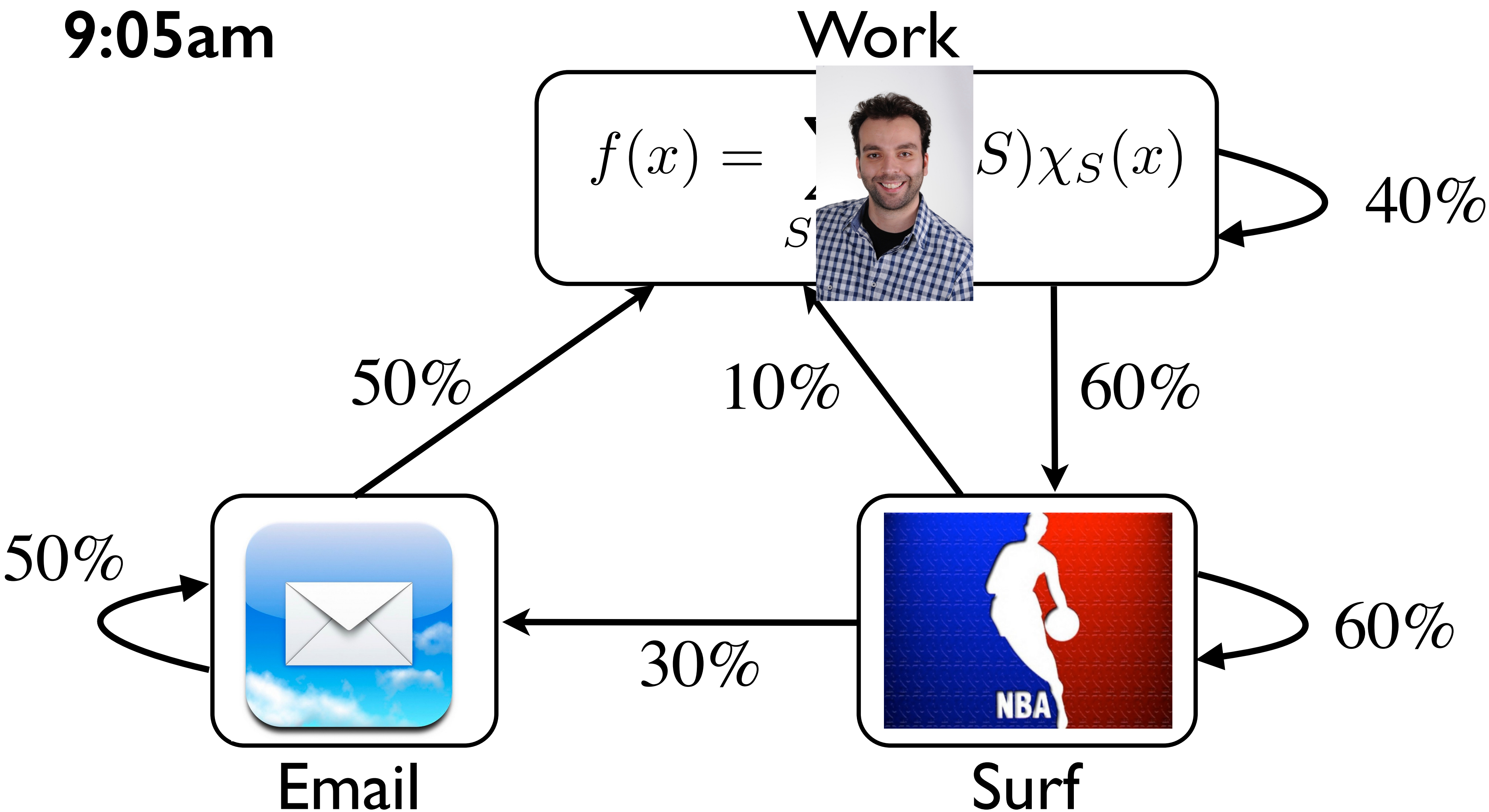
My typical day (when I was a student)

9:04am



My typical day (when I was a student)

9:05am



And now

Prepare 15-251 slides

Markov Model

Markov Model

Andrey Markov (1856 - 1922)

Russian mathematician.

Famous for his work on <u>random processes</u>.

($\Pr[X \geq c \cdot \mathbf{E}[X]] \leq 1/c$ is Markov's Inequality.)

Markov Model

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random processes.

($\Pr[X \geq c \cdot \mathbf{E}[X]] \leq 1/c$ is Markov's Inequality.)

A model for the evolution of a random system.

The future is independent of the past, given the present.

Cool things about the Markov model

- It is a very general and natural model.

  Extraordinary number of applications in many different disciplines:

    computer science, mathematics, biology, physics, chemistry, economics, psychology, music, baseball,...

- The model is simple and neat.

- A beautiful mathematical theory behind it.

  Starts simple, goes deep.

The plan

Motivating examples and applications

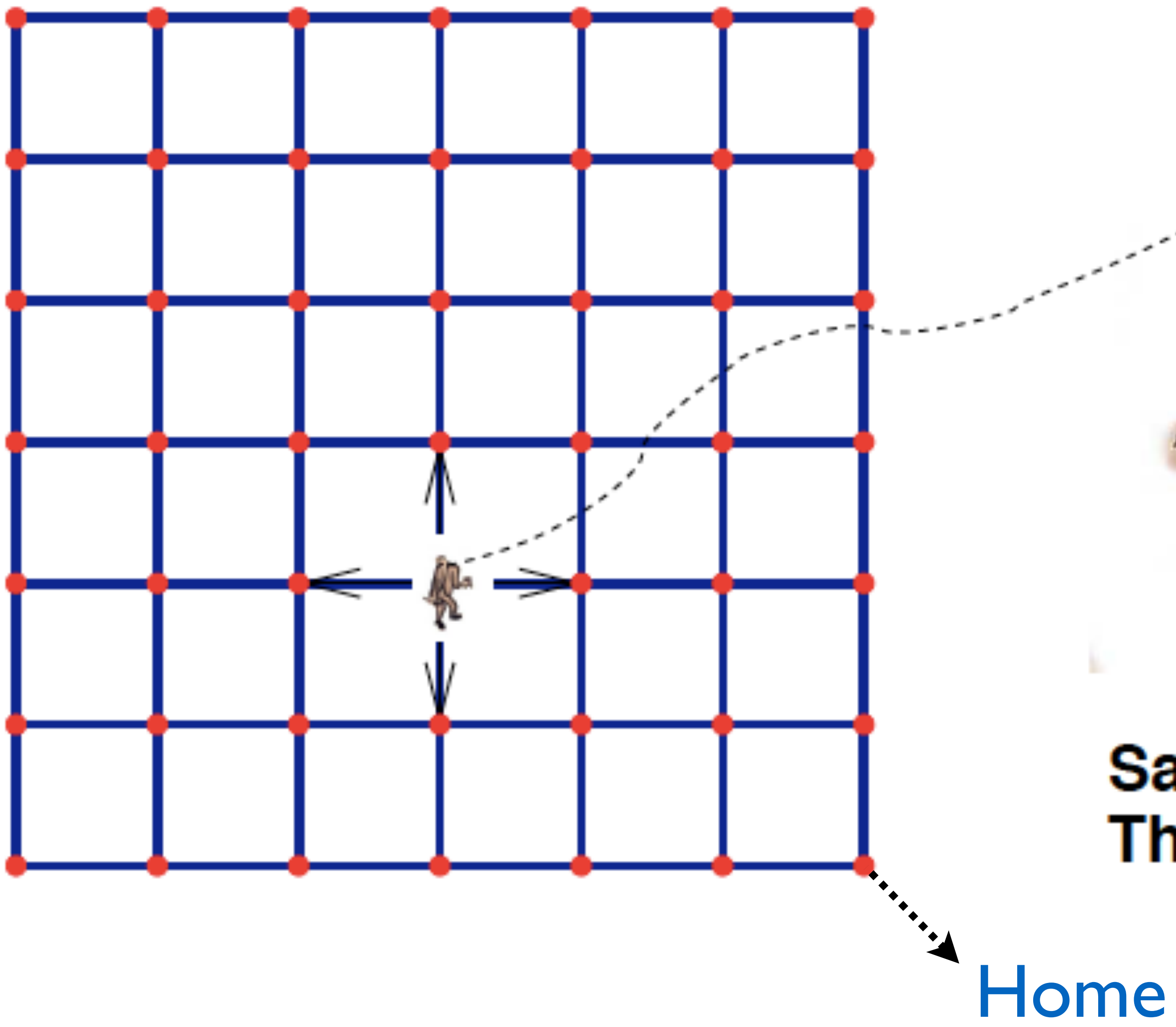
Basic mathematical representation and properties

Applications

The future is independent of the past, given the present.

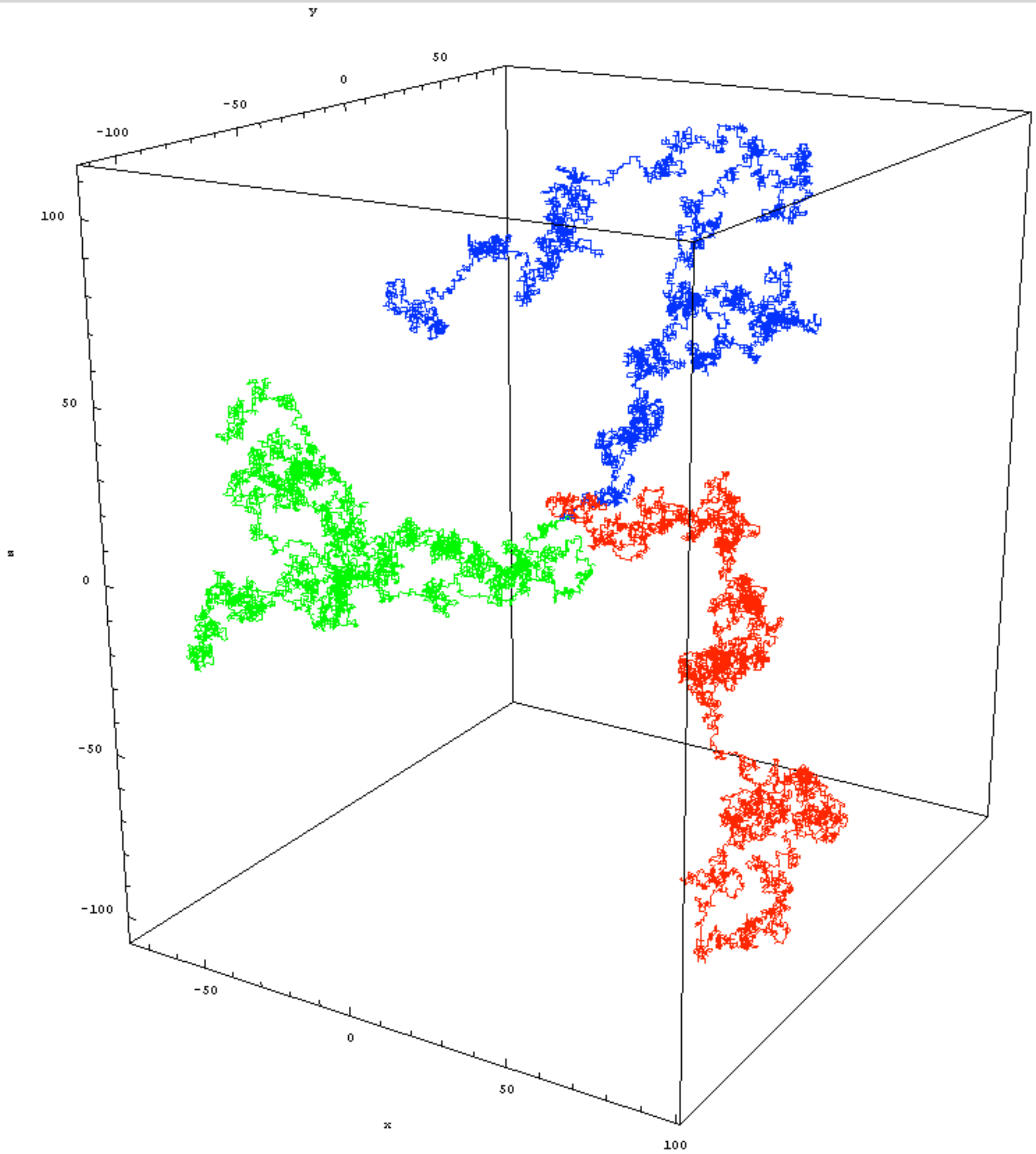
Some Examples of Markov Models

Example: Drunkard Walk



Salvador Dali (1922)
The Drunkard

Example: Diffusion Process



Example: Weather

A very(!!) simplified model for the weather.

Probabilities on a daily basis:

S = sunny
R = rainy

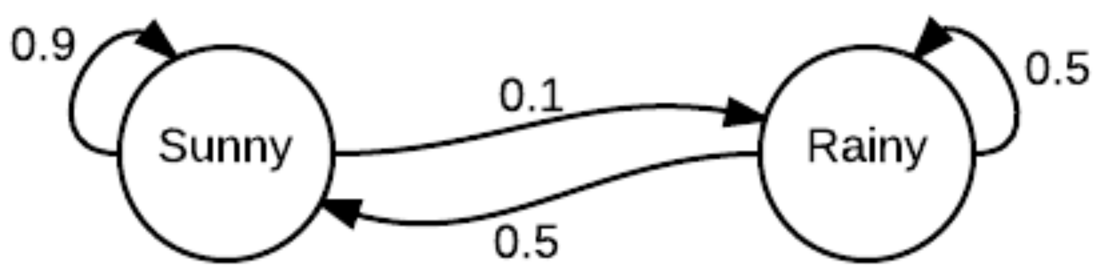
Pr[sunny to rainy] = 0.1

Pr[sunny to sunny] = 0.9

Pr[rainy to rainy] = 0.5

Pr[rainy to sunny] = 0.5

$$\begin{array}{cc} & \begin{array}{cc} \mathbf{S} & \mathbf{R} \end{array} \\ \begin{array}{c} \mathbf{S} \\ \mathbf{R} \end{array} & \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} \end{array}$$



Encode more information about current state for a more accurate model.

Example: Life Insurance

Goal of insurance company:

figure out how much to charge the clients.

Find a model for how long a client will live.

Probabilistic model of health on a monthly basis:

Pr[healthy to sick] = 0.3
Pr[sick to healthy] = 0.8
Pr[sick to death] = 0.1
Pr[healthy to death] = 0.01
Pr[healthy to healthy] = 0.69
Pr[sick to sick] = 0.1
Pr[death to death] = 1

Example: Life Insurance

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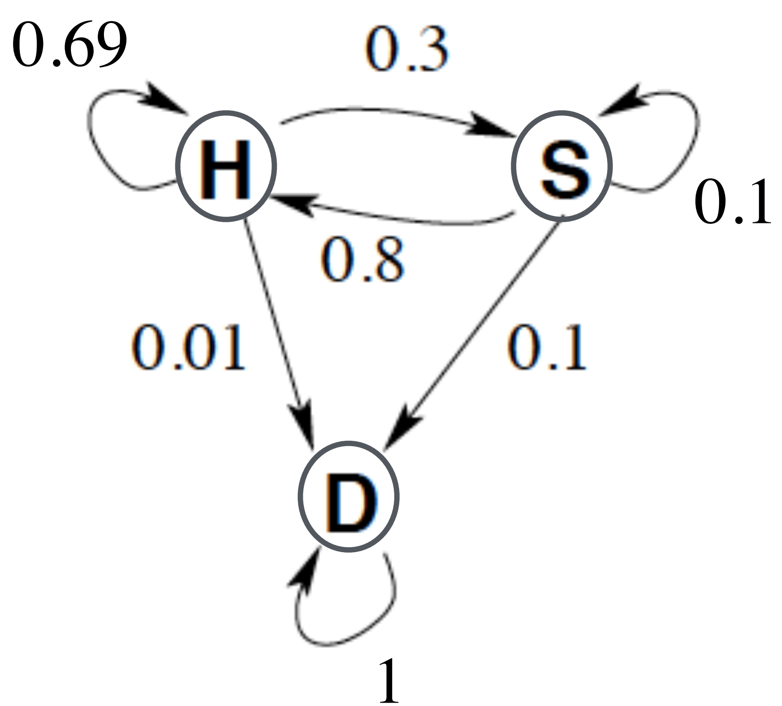
Find a model for how long a client will live.

Probabilistic model of health on a monthly basis:

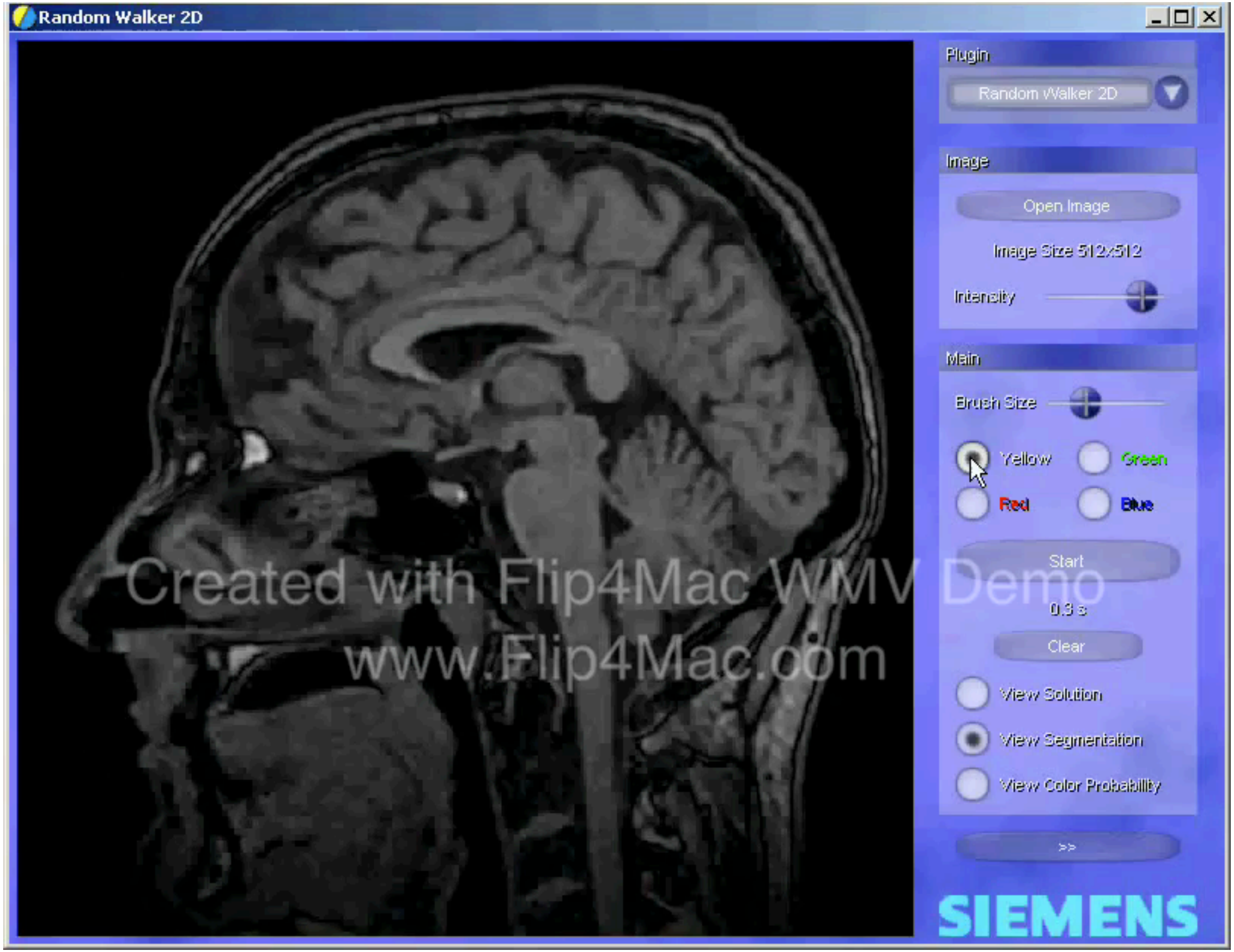


$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{H} & \mathbf{S} & \mathbf{D} \end{array} \\ \begin{array}{c} \mathbf{H} \\ \mathbf{S} \\ \mathbf{D} \end{array} & \begin{bmatrix} 0.69 & 0.3 & 0.01 \\ 0.8 & 0.1 & 0.1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Some Applications of Markov Models

Application: Algorithmic Music Composition

Application: Image Segmentation



Application: Automatic Text Generation

Random text generated by a computer
(putting random words together):

"While at a conference a few weeks back, I spent an interesting evening with a grain of salt."

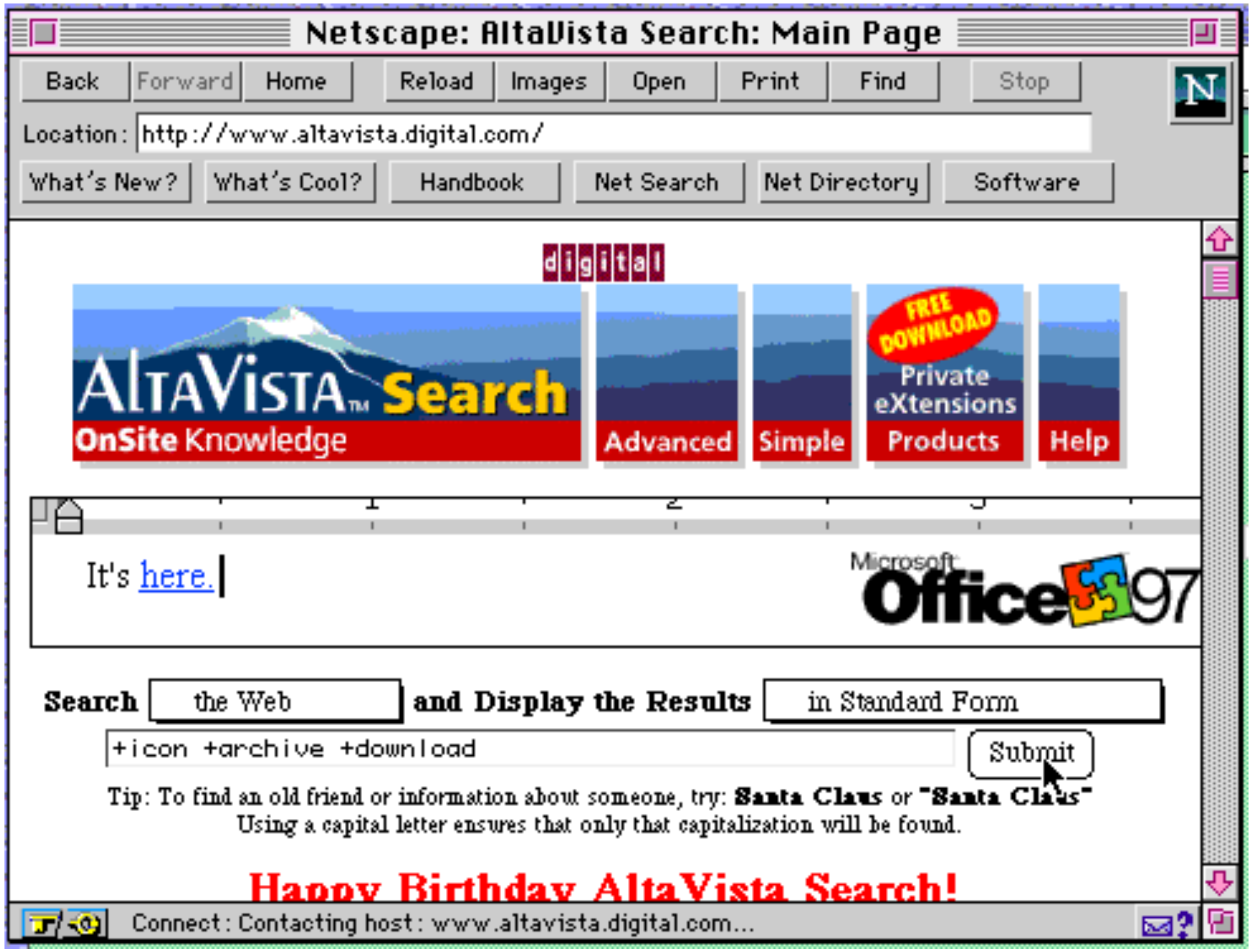
Google: Mark V Shaney

Application: Speech Recognition

Speech recognition software programs use Markov models to listen to the sound of your voice and convert it into text.

Application: Google PageRank

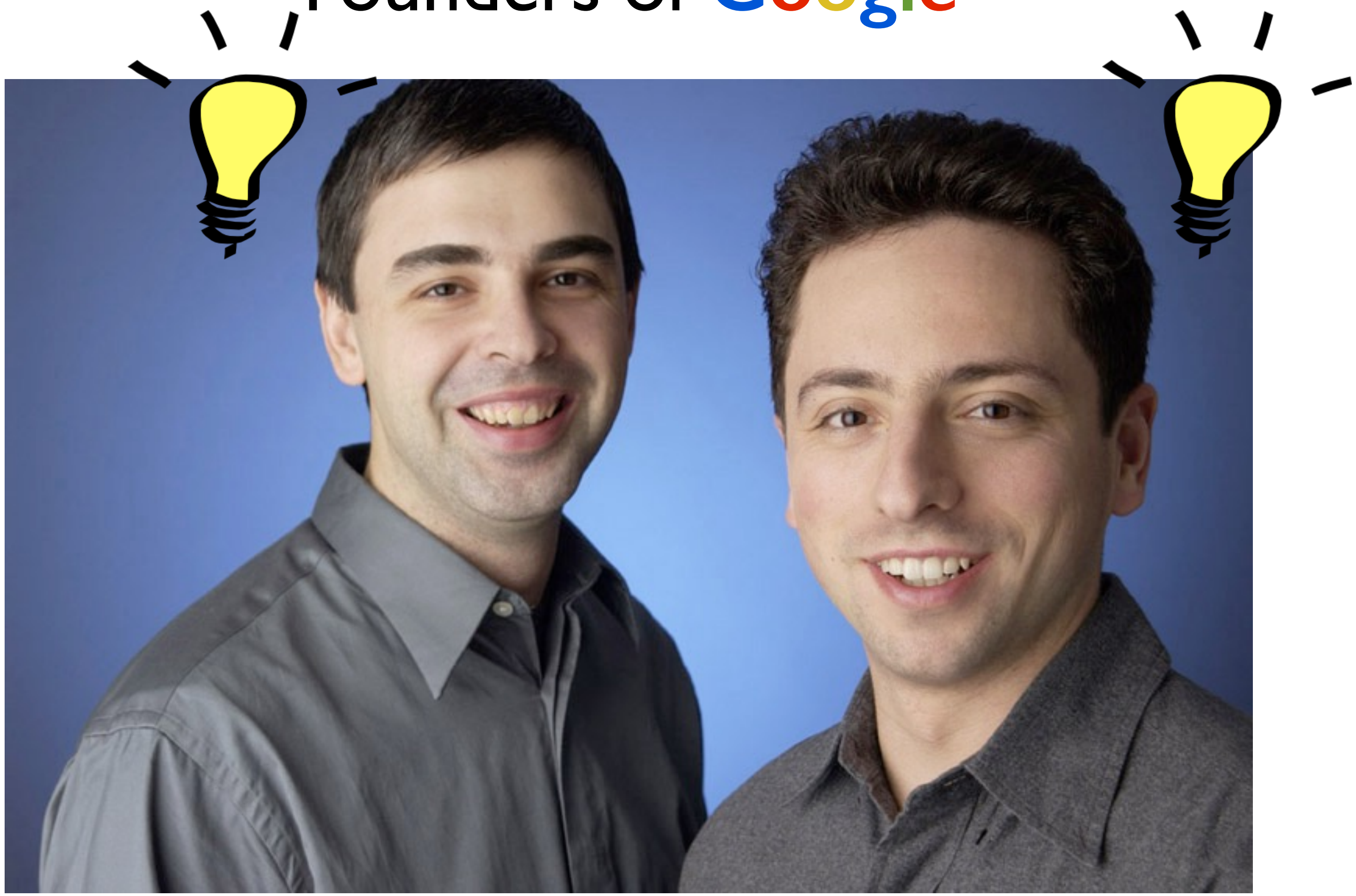
1997: Web search was horrible



Sorts webpages by number of occurrences of keyword(s).

Application: Google PageRank

Founders of Google



Larry Page Sergey Brin

$20Billionaires

Application: Google PageRank

Jon Kleinberg

Nevanlinna Prize

Application: Google PageRank

How does Google order the webpages displayed after a search?

<u>**2 important factors:**</u>

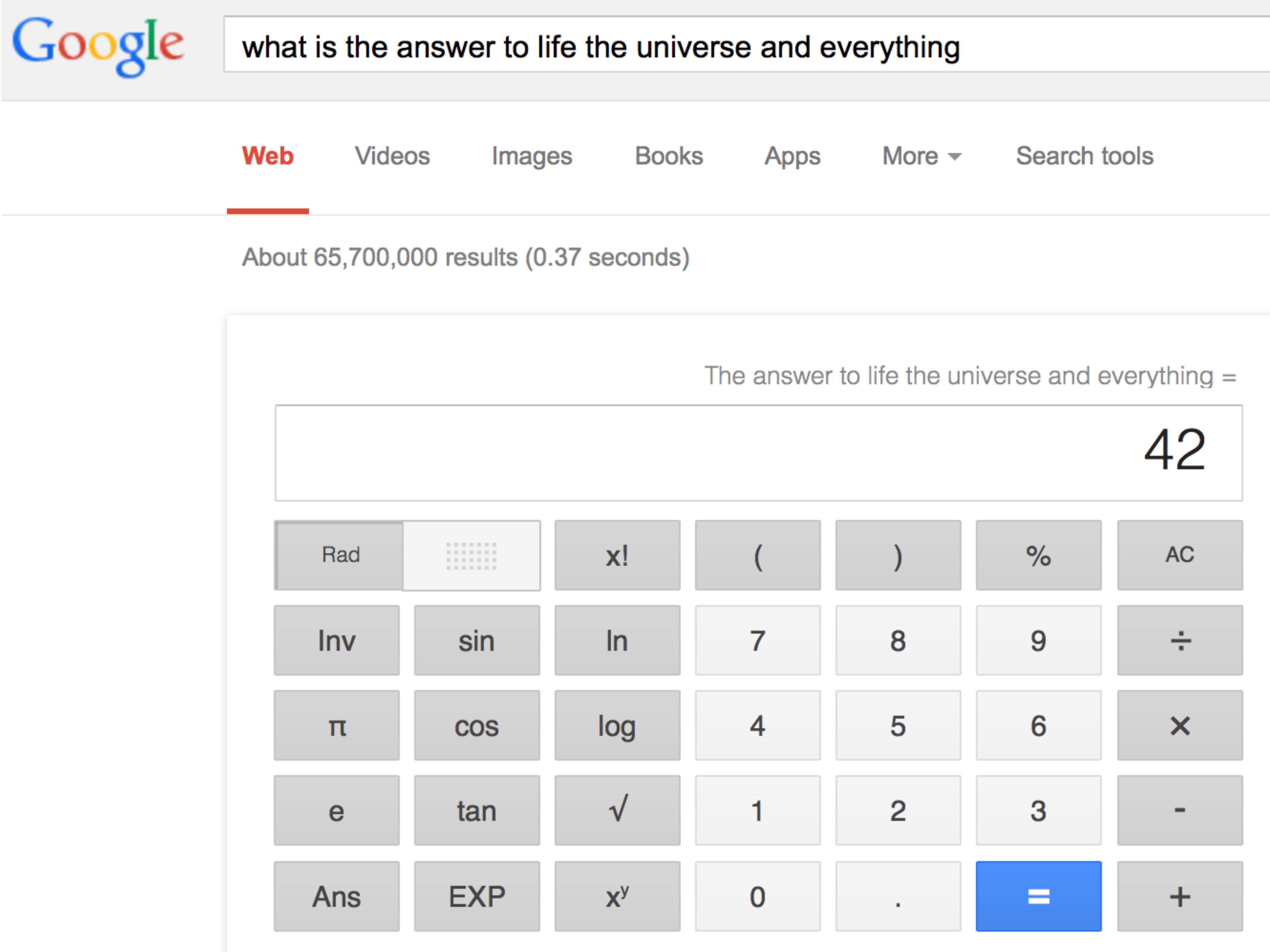
- Relevance of the page.

- <u>Reputation</u> of the page.

  The number and reputation of links pointing to that page.

  Reputation is measured using PageRank.

  PageRank is calculated using a Markov Chain.


Google
what is the answer to life the universe and everything
Web
Videos
Images
Books
Apps
More
Search tools
About 65,700,000 results (0.37 seconds)
The answer to life the universe and everything =
42
Rad
x!
(
)
%
AC
Inv
sin
ln
7
8
9
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The plan

Motivating examples and applications

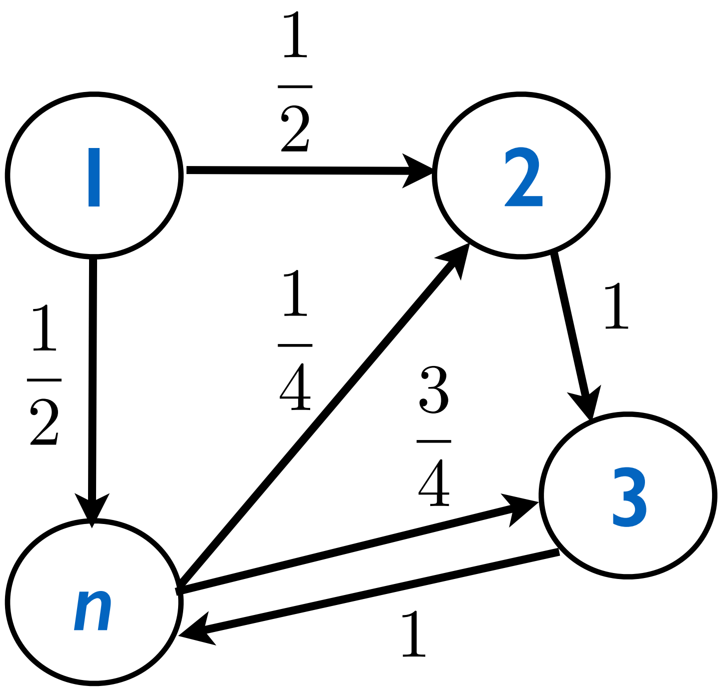
Basic mathematical representation and properties

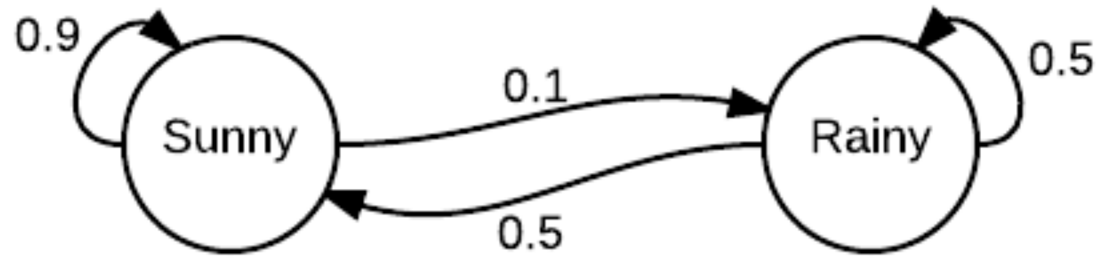
Applications

The Setting

There is a system with *n* possible states/values.

At each time step, the state changes probabilistically.





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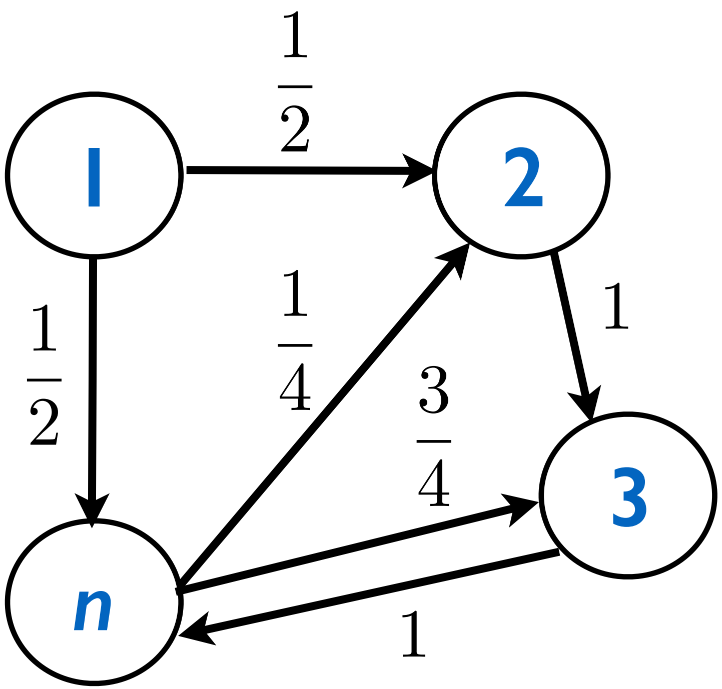
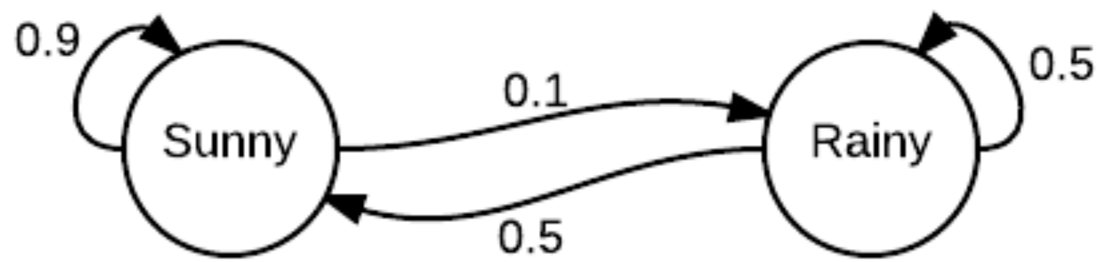




Memoryless

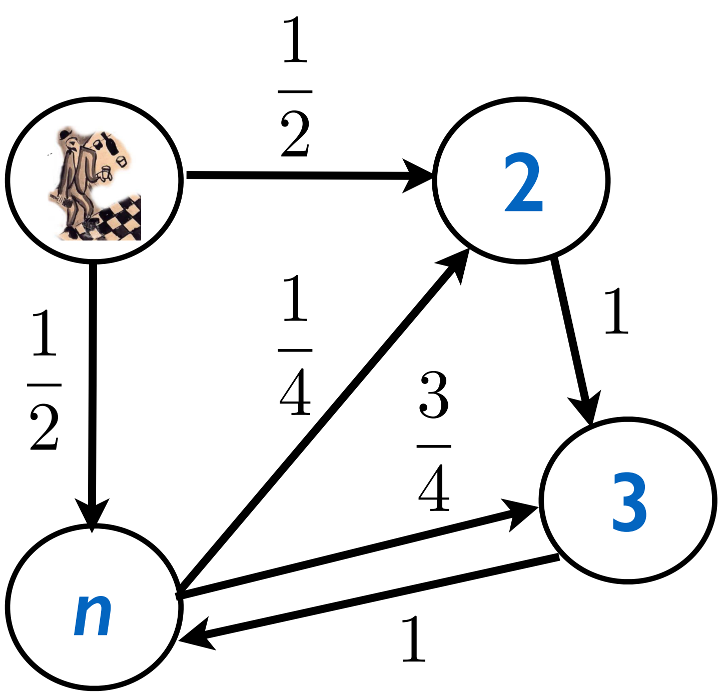
The next state only depends on the current state.

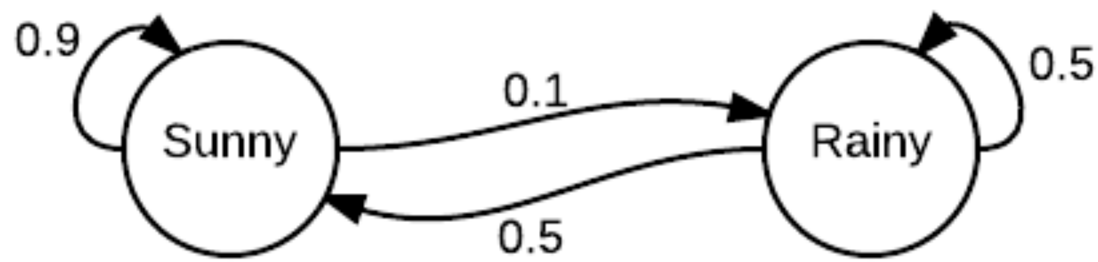
Evolution of the system: random walk on the graph.

The Setting

There is a system with *n* possible states/values.

At each time step, the state changes probabilistically.





Memoryless

The next state only depends on the current state.

Evolution of the system: random walk on the graph.

The Definition

A **Markov Chain** is a directed graph with $V = \{1, 2, \ldots, n\}$ such that:

- Each edge is labeled with a value in $(0, 1]$ (a positive probability).
  self-loops allowed
- At each vertex, the probabilities on outgoing edges sum to 1.

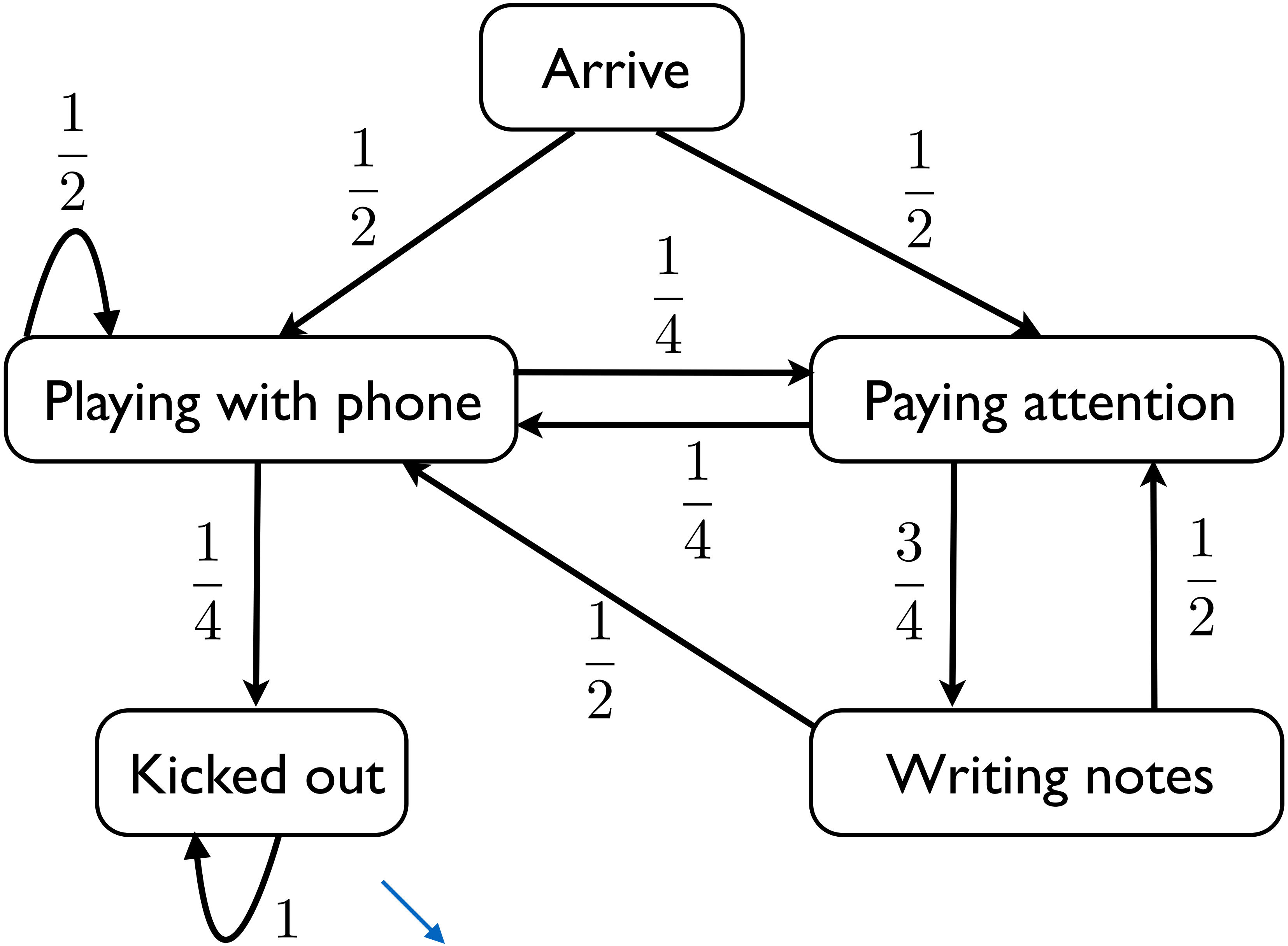
(- We usually assume the graph is strongly connected.
i.e. there is a path from *i* to *j* for any *i* and *j*.)

The vertices of the graph are called states.

The edges are called transitions.

The label of an edge is a transition probability.

Example: Markov Chain for a Lecture



This is not strongly connected.

Notation

Given some Markov Chain with n states:

For each $t = 0, 1, 2, 3, \ldots$ we have a random variable:

$X_t =$ the state we are in after t steps.

Define $\pi_t[i] = \Pr[X_t = i]$.

$$\pi_t = \begin{array}{c} \begin{array}{cccc} 1 & 2 & & n \end{array} \\ [p_1 \; p_2 \; \cdots p_n] \end{array}$$

$\pi_t[i] =$ probability of being in state *i* after t steps.

$$\sum_i p_i = 1$$

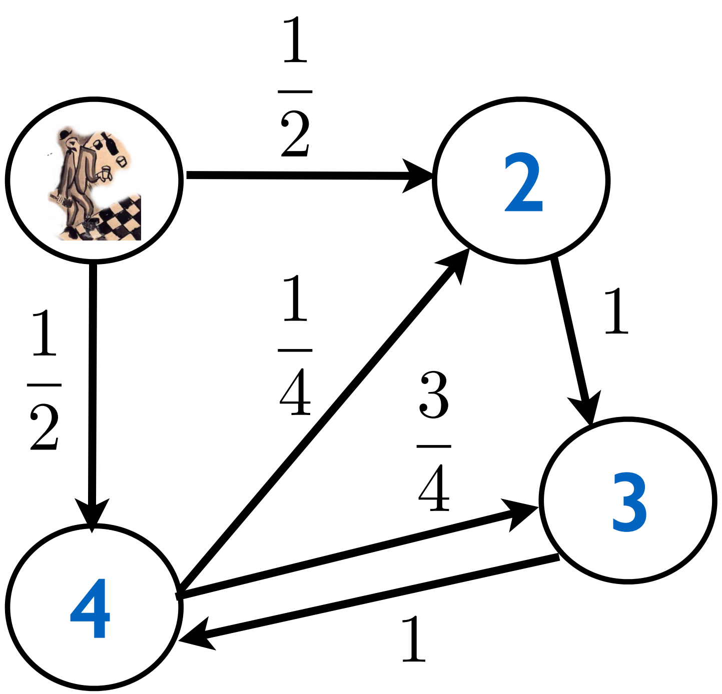
We write $X_t \sim \pi_t$. (X_t has distribution π_t)

Note that someone has to provide π_0.

Once this is known, we get the distributions $\pi_1, \pi_2, \ldots$

Notation

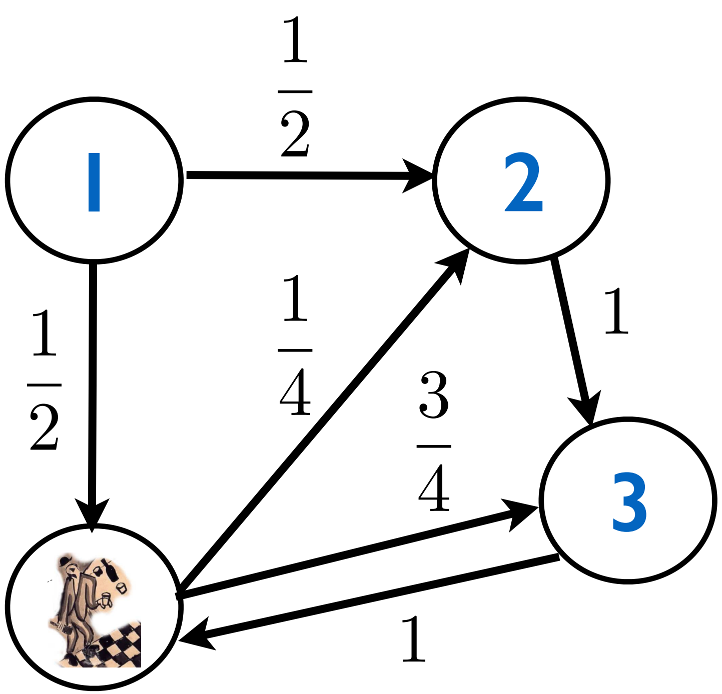
Let's say we start at state 1, i.e., $X_0 \sim \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ [1 \quad 0 \quad 0 \quad 0] \end{array} = \pi_0$



$X_0 = 1 \qquad X_0 \sim \pi_0$

Notation

Let's say we start at state 1, i.e., $X_0 \sim \begin{array}{c} \begin{matrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{matrix} \\ [\begin{matrix} 1 & 0 & 0 & 0 \end{matrix}] \end{array} = \pi_0$

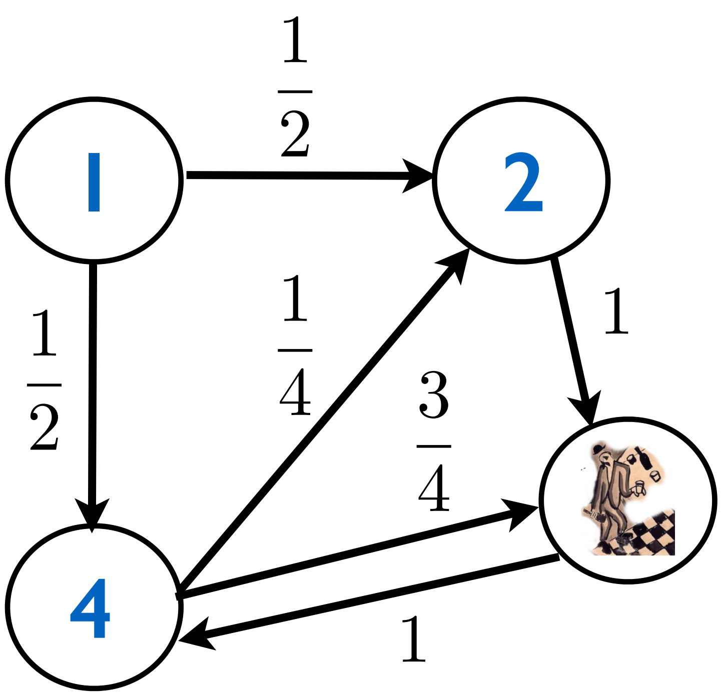


$$X_0 = 1 \qquad X_0 \sim \pi_0$$

$$X_1 = 4 \qquad X_1 \sim \pi_1$$

Notation

Let's say we start at state 1, i.e., $X_0 \sim \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \end{array}$ $[1 \quad 0 \quad 0 \quad 0] = \pi_0$



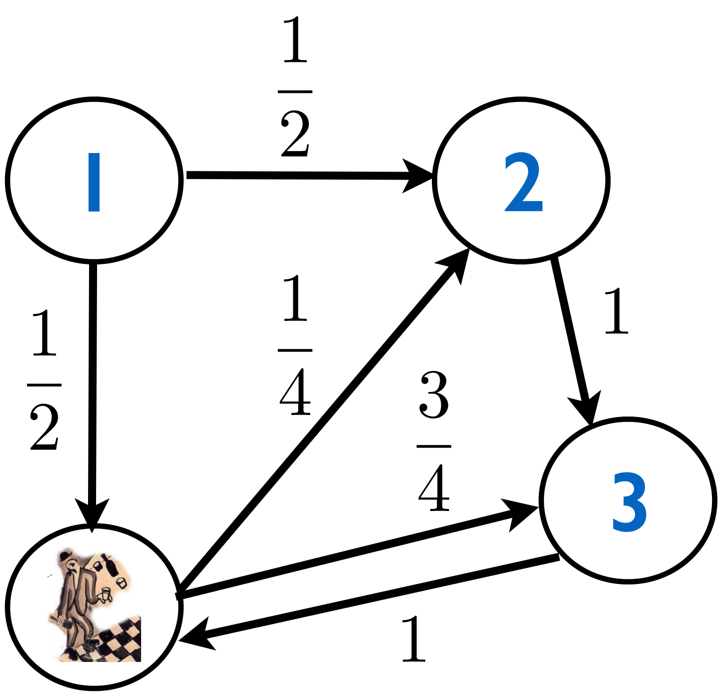
$X_0 = 1$ $\quad X_0 \sim \pi_0$

$X_1 = 4$ $\quad X_1 \sim \pi_1$

$X_2 = 3$ $\quad X_2 \sim \pi_2$

Notation

Let's say we start at state 1, i.e., $X_0 \sim \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ [1 \quad 0 \quad 0 \quad 0] \end{array} = \pi_0$



$$X_0 = 1 \qquad X_0 \sim \pi_0$$

$$X_1 = 4 \qquad X_1 \sim \pi_1$$

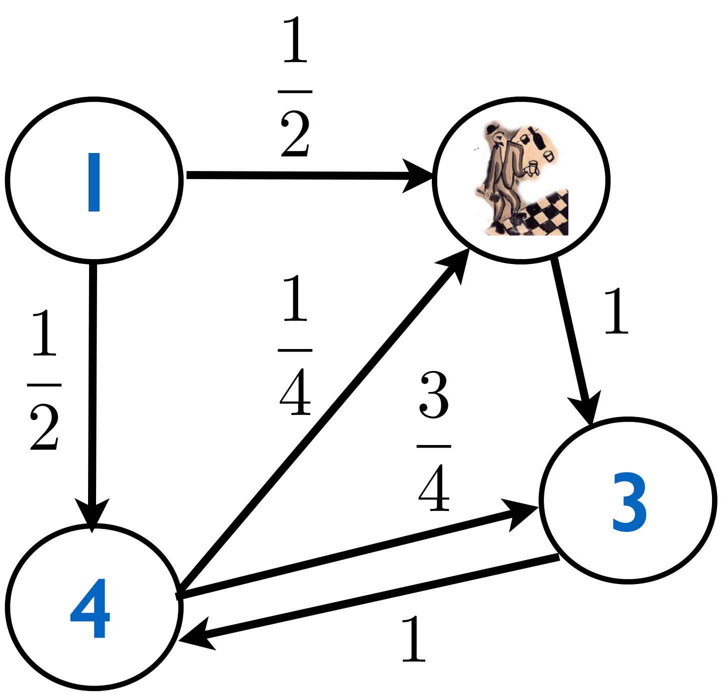
$$X_2 = 3 \qquad X_2 \sim \pi_2$$

$$X_3 = 4 \qquad X_3 \sim \pi_3$$

Notation

Let's say we start at state 1, i.e., $X_0 \sim \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline \end{array}$

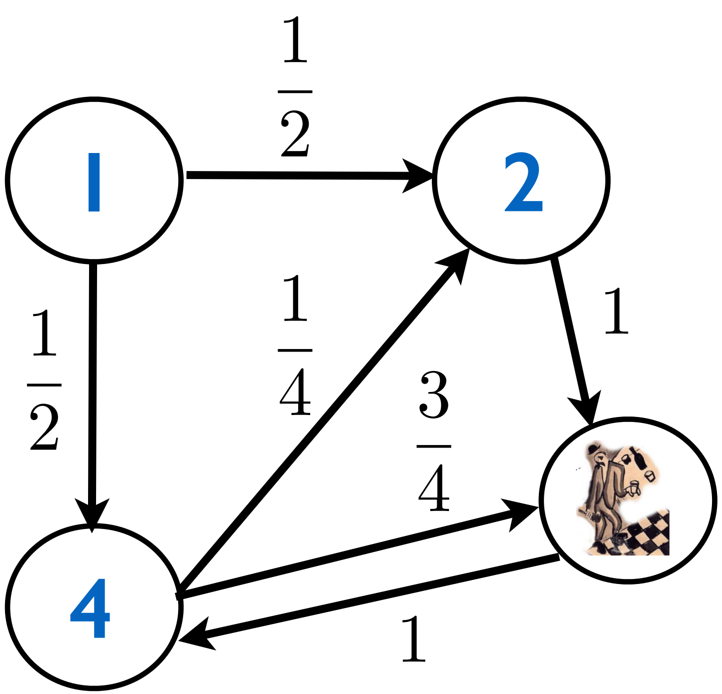
$$X_0 \sim \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ [1 & 0 & 0 & 0] \end{array} = \pi_0$$



$$\begin{aligned} X_0 &= 1 & X_0 &\sim \pi_0 \\ X_1 &= 4 & X_1 &\sim \pi_1 \\ X_2 &= 3 & X_2 &\sim \pi_2 \\ X_3 &= 4 & X_3 &\sim \pi_3 \\ X_4 &= 2 & X_4 &\sim \pi_4 \end{aligned}$$

Notation

Let's say we start at state 1, i.e., $X_0 \sim \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \end{array}$ $\begin{array}{c} \\ [1 \quad 0 \quad 0 \quad 0] = \pi_0 \end{array}$



$$X_0 = 1 \qquad X_0 \sim \pi_0$$

$$X_1 = 4 \qquad X_1 \sim \pi_1$$

$$X_2 = 3 \qquad X_2 \sim \pi_2$$

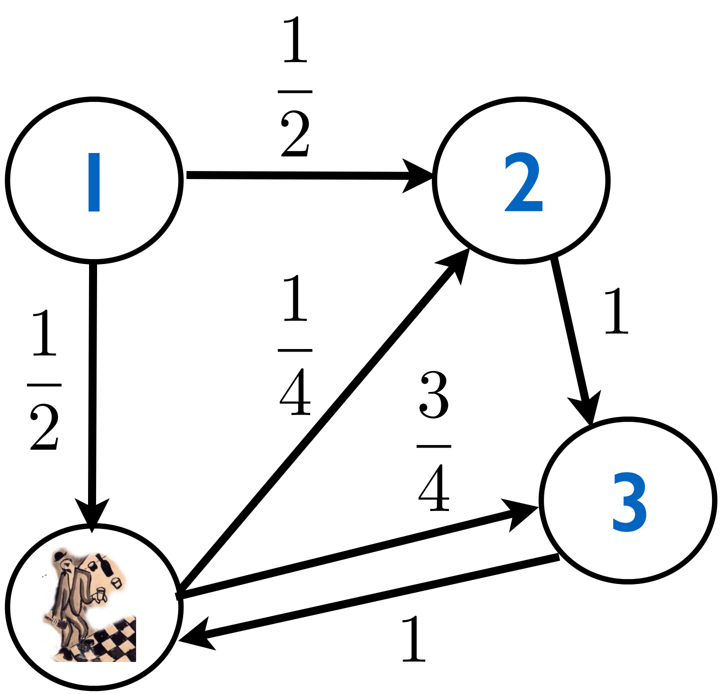
$$X_3 = 4 \qquad X_3 \sim \pi_3$$

$$X_4 = 2 \qquad X_4 \sim \pi_4$$

$$X_5 = 3 \qquad X_5 \sim \pi_5$$

Notation

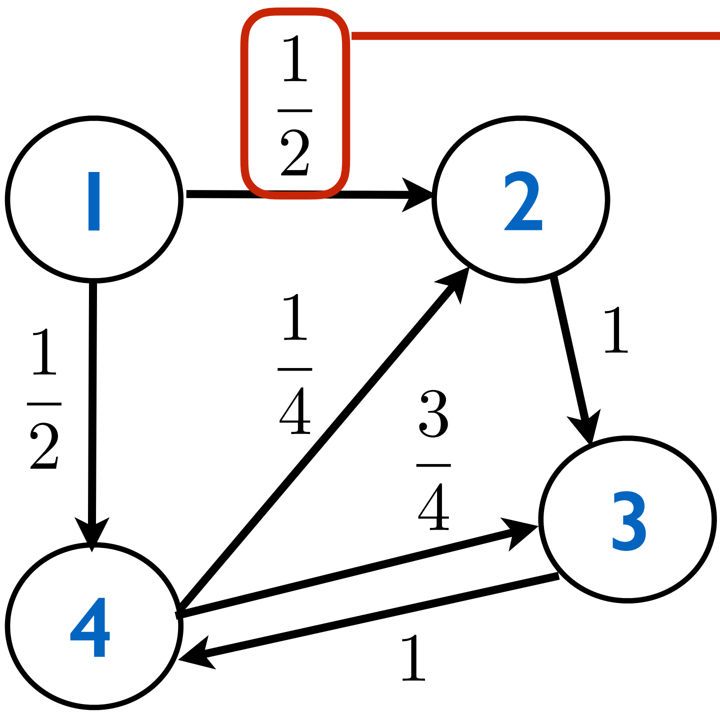
Let's say we start at state 1, i.e., $X_0 \sim \begin{array}{cccc} \color{blue}{1} & \color{blue}{2} & \color{blue}{3} & \color{blue}{4} \\ \end{array}$ $\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right] = \pi_0$



$$\begin{array}{ll} X_0 = 1 & X_0 \sim \pi_0 \\ X_1 = 4 & X_1 \sim \pi_1 \\ X_2 = 3 & X_2 \sim \pi_2 \\ X_3 = 4 & X_3 \sim \pi_3 \\ X_4 = 2 & X_4 \sim \pi_4 \\ X_5 = 3 & X_5 \sim \pi_5 \\ X_6 = 4 & X_6 \sim \pi_6 \\ \vdots & \end{array}$$

Notation

Let's say we start at state 1, i.e., $X_0 \sim \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ [1 \quad 0 \quad 0 \quad 0] \end{array} = \pi_0$



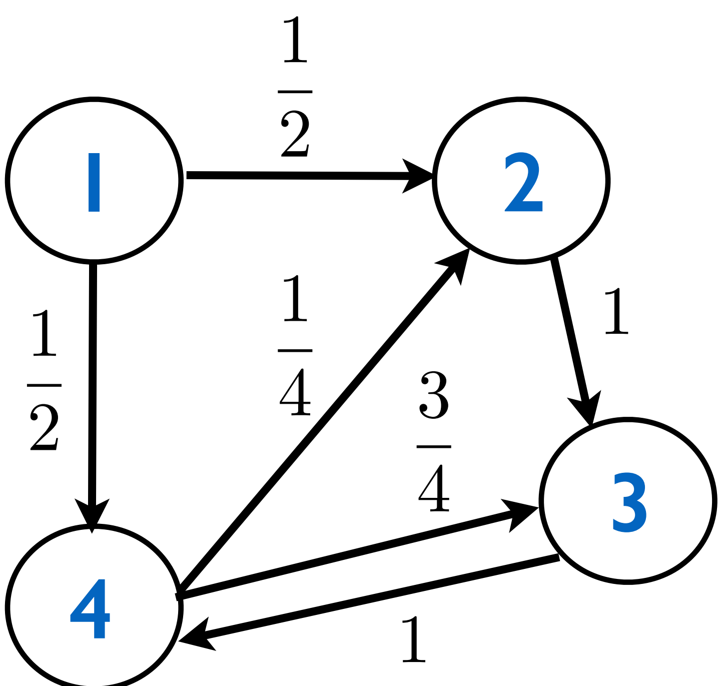
$$\Pr[1 \rightarrow 2 \text{ in one step}]$$

$$= \Pr[X_1 = 2 \mid X_0 = 1]$$

$$= \Pr[X_t = 2 \mid X_{t-1} = 1]$$

Notation

Let's say we start at state 1, i.e., $X_0 \sim \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \end{array}$ $[1 \quad 0 \quad 0 \quad 0] = \pi_0$



$$\Pr[X_1 = 2 | X_0 = 1] = \frac{1}{2}$$

$$\Pr[X_1 = 3 | X_0 = 1] = 0$$

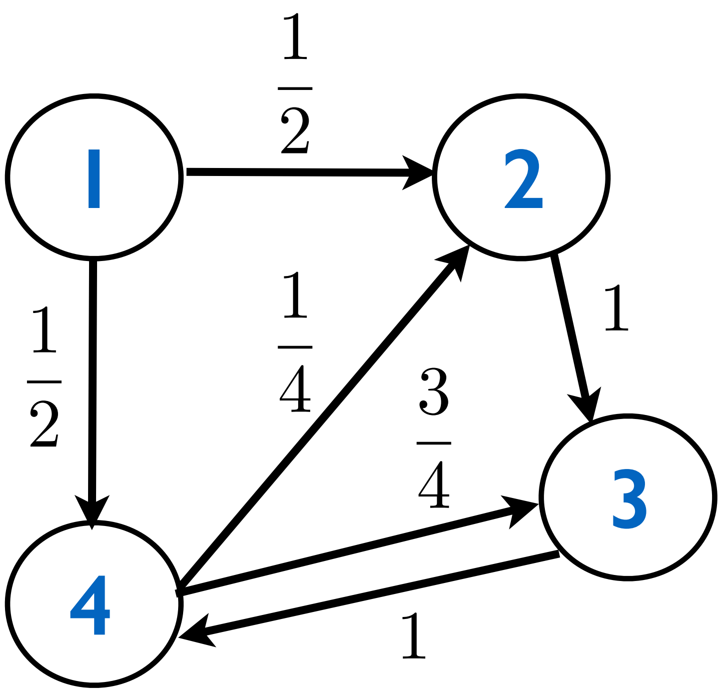
$$\Pr[X_1 = 4 | X_0 = 1] = \frac{1}{2}$$

$$\Pr[X_1 = 1 | X_0 = 1] = 0$$

$$\forall t \quad \Pr[X_t = 2 | X_{t-1} = 4] = \frac{1}{4}$$

$$\forall t \quad \Pr[X_t = 3 | X_{t-1} = 2] = 1$$

Notation



$$\begin{array}{c|c} & \begin{matrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{matrix} \\ \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{matrix} & \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} \end{array}$$

Transition Matrix

A Markov Chain with **n** states
can be characterized by the **n x n** **transition matrix** K :

$$\forall i, j \in \{1, 2, \ldots, n\} \quad K[i,j] = \Pr[X_t = j \mid X_{t-1} = i]$$
$$= \Pr[i \to j \text{ in one step}]$$

<u>Note</u>: rows of K sum to 1.

Some Fundamental and Natural Questions

What is the probability of being in state *i* after ***t*** steps (given some initial state)?

$$\pi_t[i] = ?$$

What is the expected time of reaching state *i* when starting at state *j* ?

What is the expected time of having visited every state (given some initial state)?

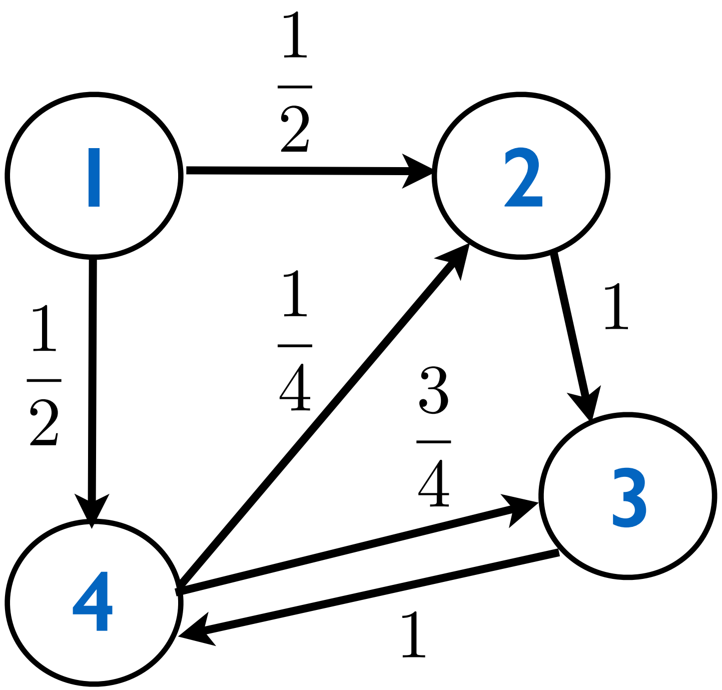
$$\vdots$$

How do you answer such questions?

Mathematical representation of the evolution

Suppose we start at state 1 and let the system evolve.

How can we mathematically represent the evolution?

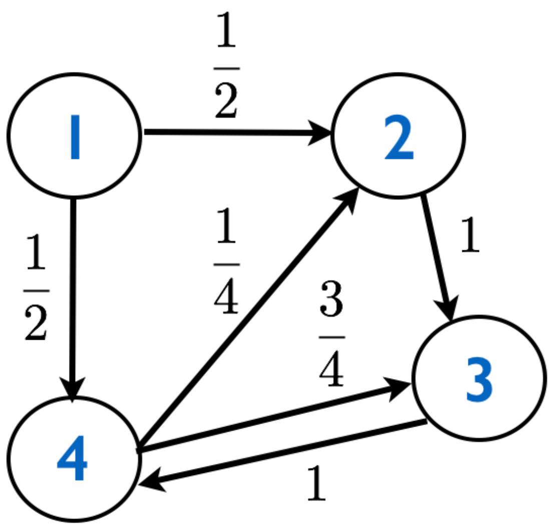


$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} \end{array}$$

$$\pi_0 = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

What is π_1? By inspection, $\pi_1 = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \end{array}$.

Poll



$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} \end{array}$$

Given $\pi_1 = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \end{array}$, what is π_2 ?

$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}$

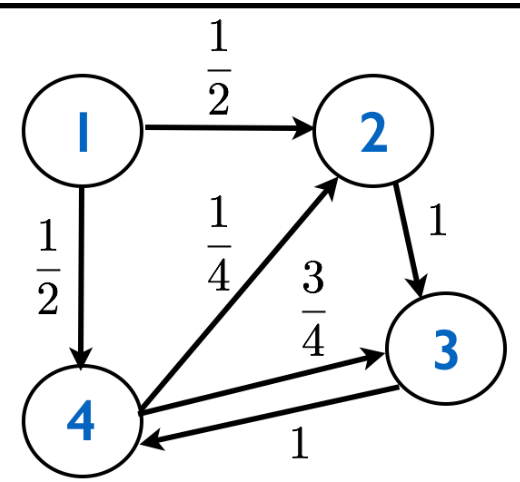
$\begin{bmatrix} 0 & \frac{5}{8} & \frac{3}{8} & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

$\begin{bmatrix} 0 & \frac{1}{8} & \frac{7}{8} & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$

Mathematical representation of the evolution



$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} & & \\ 3 & & & & \\ 4 & & & & \end{array}$$

$$\pi_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

What is π_1? $\quad \pi_1[j] = \Pr[X_1 = j]$

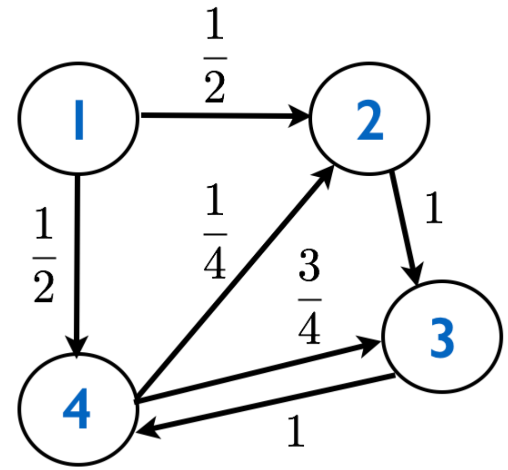
(law of total probability) $= \sum_{i=1}^{4} \Pr[X_1 = j \mid X_0 = i] \Pr[X_0 = i]$

$$= \sum_{i=1}^{4} K[i,j] \cdot \pi_0[i] = (\pi_0 \cdot K)[j]$$

matrix mult.

This is true for any j.

Mathematical representation of the evolution



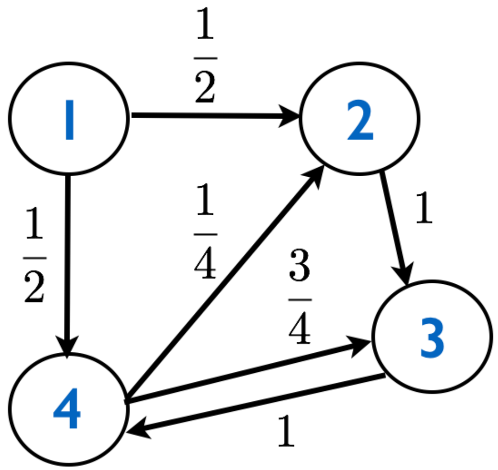
$$\begin{array}{c|cccc} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \hline \mathbf{1} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \mathbf{2} & 0 & 0 & 1 & 0 \\ \mathbf{3} & 0 & 0 & 0 & 1 \\ \mathbf{4} & 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{array}$$

The probability of states after 1 step:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}}_{\pi_0} \underbrace{\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}}_{K} = \underbrace{\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}}_{\pi_1}$$

the new state (probabilistic)

Mathematical representation of the evolution



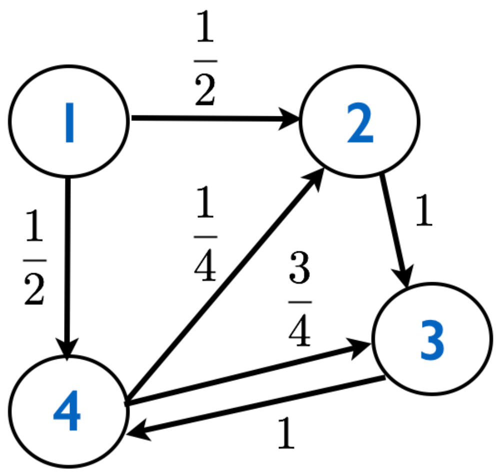
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} \end{array}$$

The probability of states after 2 steps:

$$\underset{\pi_1}{\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}} \underset{K}{\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}} = \underset{\pi_2}{\begin{bmatrix} 0 & \frac{1}{8} & \frac{7}{8} & 0 \end{bmatrix}}$$

the new state (probabilistic)

Mathematical representation of the evolution



$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} \end{array}$$

$$\pi_1 = \pi_0 \cdot K$$

$$\pi_2 = \pi_1 \cdot K$$

So $$\pi_2 = (\pi_0 \cdot K) \cdot K$$
$$= \pi_0 \cdot K^2$$

Mathematical representation of the evolution

In general:

If the initial probabilistic state is $\begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} = \pi_0$

$p_i =$ probability of being in state *i*,

$p_1 + p_2 + \cdots + p_n = 1$,

after ***t*** steps, the probabilistic state is:

$$\begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \begin{bmatrix} \text{Transition} \\ \text{Matrix} \\ \\ \end{bmatrix}^{t} = \pi_t$$

Remarkable Property of Markov Chains

What happens in the long run?

i.e., can we say anything about π_t for large t ?

Suppose the Markov chain is "aperiodic".

Then, as the system evolves, the probabilistic state ***converges*** to a limiting probabilistic state.

As $t \to \infty$, for any $\pi_0 = [p_1 \quad p_2 \quad \cdots \quad p_n]$:

$$[p_1 \quad p_2 \quad \cdots \quad p_n] \begin{bmatrix} \text{Transition} \\ \text{Matrix} \end{bmatrix}^t \longrightarrow \pi$$

Remarkable Property of Markov Chains

In other words:

$$\pi_t \to \pi \quad \text{as} \quad t \to \infty \,.$$

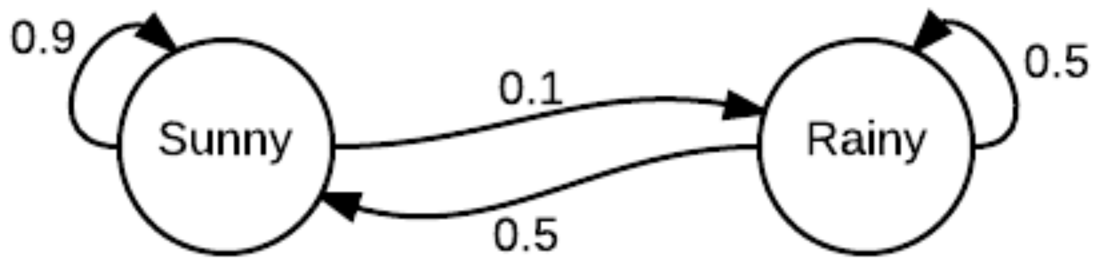
Note:

$$\pi \begin{bmatrix} \text{Transition} \\ \text{Matrix} \\ \\ \end{bmatrix} = \pi$$

stationary/invariant distribution

This π is unique.

Remarkable Property of Markov Chains



Stationary distribution is $\begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix}$.

$$\begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

In the long run, it is sunny 5/6 of the time,
it is rainy 1/6 of the time.

Remarkable Property of Markov Chains

How did I find the stationary distribution?

$$\begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}^2 = \begin{bmatrix} 0.86 & 0.14 \\ 0.7 & 0.3 \end{bmatrix}$$

$$\begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}^4 = \begin{bmatrix} 0.8376 & 0.1624 \\ 0.812 & 0.188 \end{bmatrix}$$

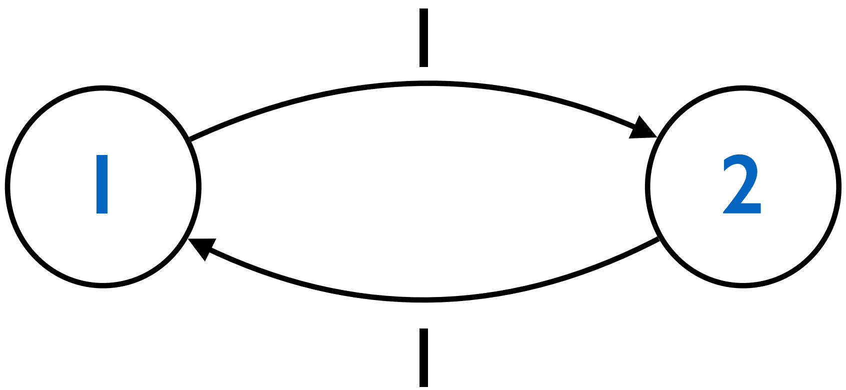
$$\begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}^8 = \begin{bmatrix} 0.833443 & 0.166557 \\ 0.832787 & 0.167213 \end{bmatrix}$$

Exercise: Why do the rows converge to π ?

Remarkable Property of Markov Chains

We needed the Markov chain to be "aperiodic".

What is a "periodic" Markov chain?



$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\pi_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$\pi_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$\pi_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$\pi_3 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$\vdots$$

There is still a *stationary* distribution.

$$\pi = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$$

But it is not a *limiting* distribution.

Summary so far

Markov Chains can be characterized by the transition matrix K.

$$K[i,j] = \Pr[X_t = j \mid X_{t-1} = i]$$
$$= \Pr[i \to j \text{ in one step}]$$

What is the probability of being in state *i* after *t* steps?

$$\pi_t = \pi_0 \cdot K^t \qquad\qquad \pi_t[i] = (\pi_0 \cdot K^t)[i]$$

There is a unique invariant distribution π : $\pi = \pi \cdot K$

For aperiodic Markov Chains: $\pi_t \to \pi$ as $t \to \infty$.

The plan

Motivating examples and applications

Basic mathematical representation and properties

Applications

How are Markov Chains applied ?

<u>2 common types of applications:</u>

1. Build a Markov chain as a statistical model of a real-world process.

   Use the Markov chain to simulate the process.

   e.g. text generation, music composition.

2. Use a measure associated with a Markov chain to approximate a quantity of interest.

   e.g. Google PageRank, image segmentation

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Automatic Text Generation

Generate a superficially real-looking text given a sample document.

Idea:

From the sample document, create a Markov chain.

Use a random walk on the Markov chain to generate text.

Example:

Collect speeches of Obama, create a Markov chain.

Use a random walk to generate new speeches.

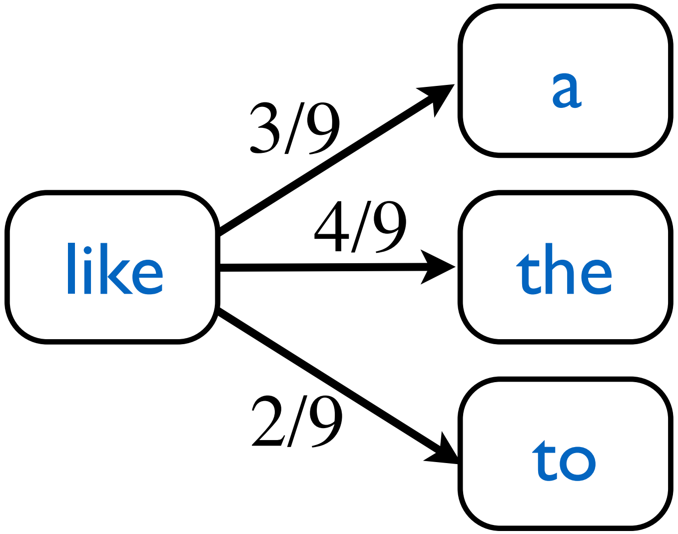
Automatic Text Generation

<u>The Markov Chain:</u>

1. For each word in the document, create a node/state.

2. Put an edge word1 ---> word2
if there is a sentence in which word2 comes after word1.

3. Edge probabilities reflect frequency of the pair of words.



like a	3 times
like the	4 times
like to	2 times

Automatic Text Generation

"I jumped up. I don't know what's going on so I am coming down with a road to opportunity. I believe we can agree on or do about the major challenges facing our country."

Automatic Text Generation

Another use:

Build a Markov chain based on speeches of Obama.

Build a Markov chain based on speeches of Bush.

Given a new quote, can predict if it is by Obama or Bush.

(by testing which Markov model the quote fits best)

Image Segmentation

Simple version

Given an image that contains an object, figure out:

which pixels correspond to the object,

which pixels correspond to the background.

i.e., label each pixel "object" or "background"

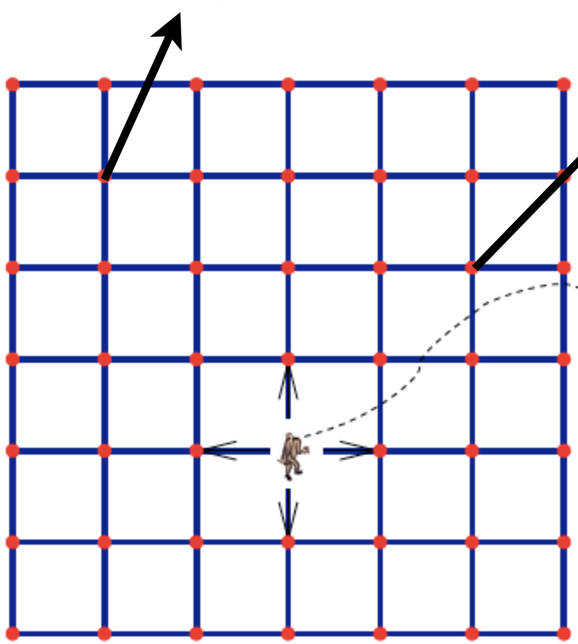
(user labels a small number of pixels with known labels)

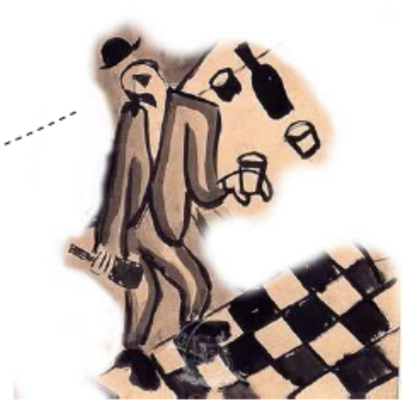
Image Segmentation

The Markov Chain:

1. Each pixel is a node/state.
2. There is an edge between adjacent pixels.
3. Edge probabilities reflect similarity between pixels.

"background" "object"





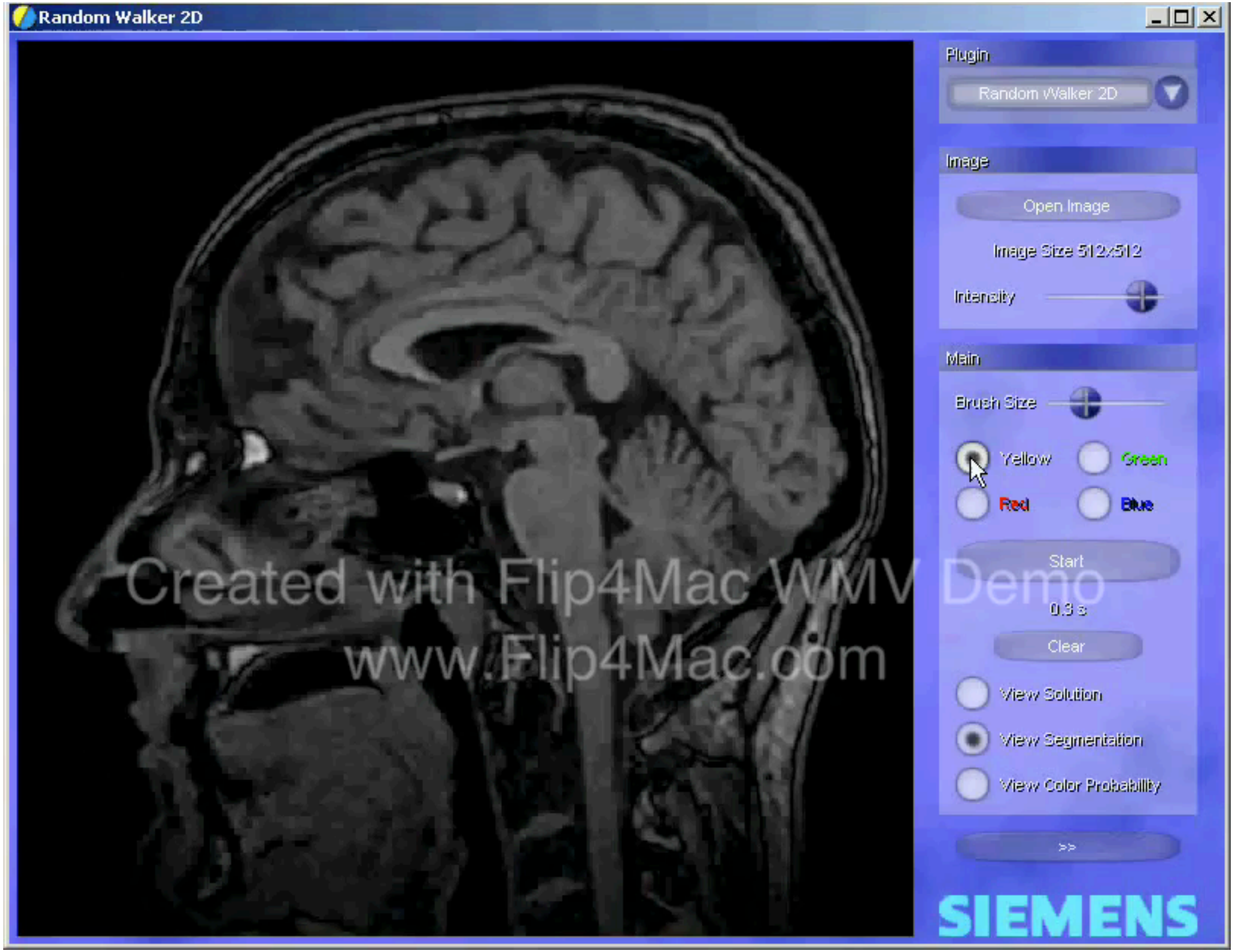
Salvador Dali (1922)
The Drunkard

Which one is more likely:

random walker first visits

"background"
or
"object"?

Image Segmentation

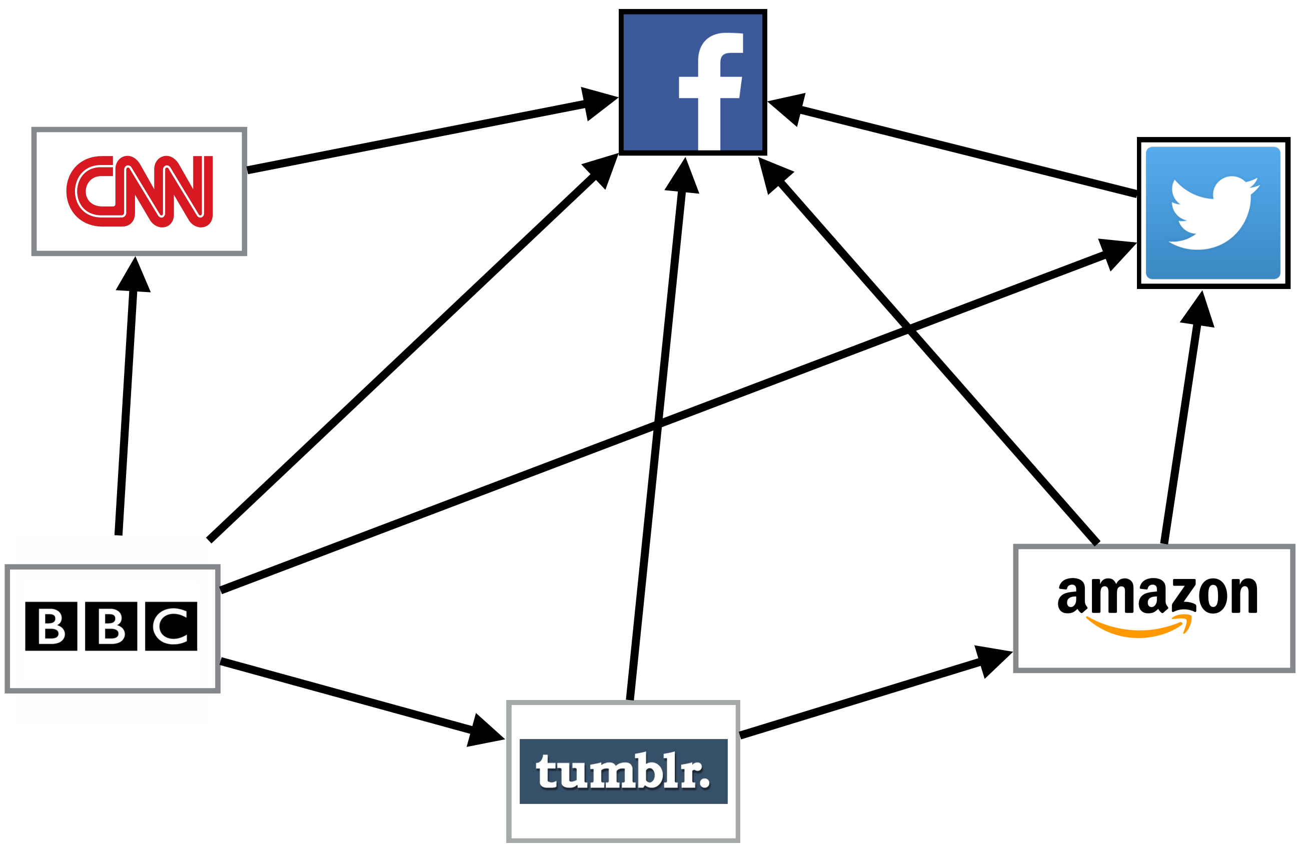


Google PageRank

PageRank is a measure of reputation:

The number and reputation of links pointing to you.

<u>**The Markov Chain:**</u>



Google PageRank

PageRank is a measure of reputation:

The number and reputation of links pointing to you.

The Markov Chain:

1. Every webpage is a node/state.

2. Each hyperlink is an edge:

   if webpage **A** has a link to webpage **B**, **A** ---> **B**

3a. If **A** has m outgoing edges, each gets label $1/m$.

3b. If **A** has no outgoing edges, put edge **A** ---> **B** $\forall$ **B**
(jump to a random page)

Google PageRank

A little tweak:

Random surfer jumps to a random page with 15% prob.

Stationary distribution:

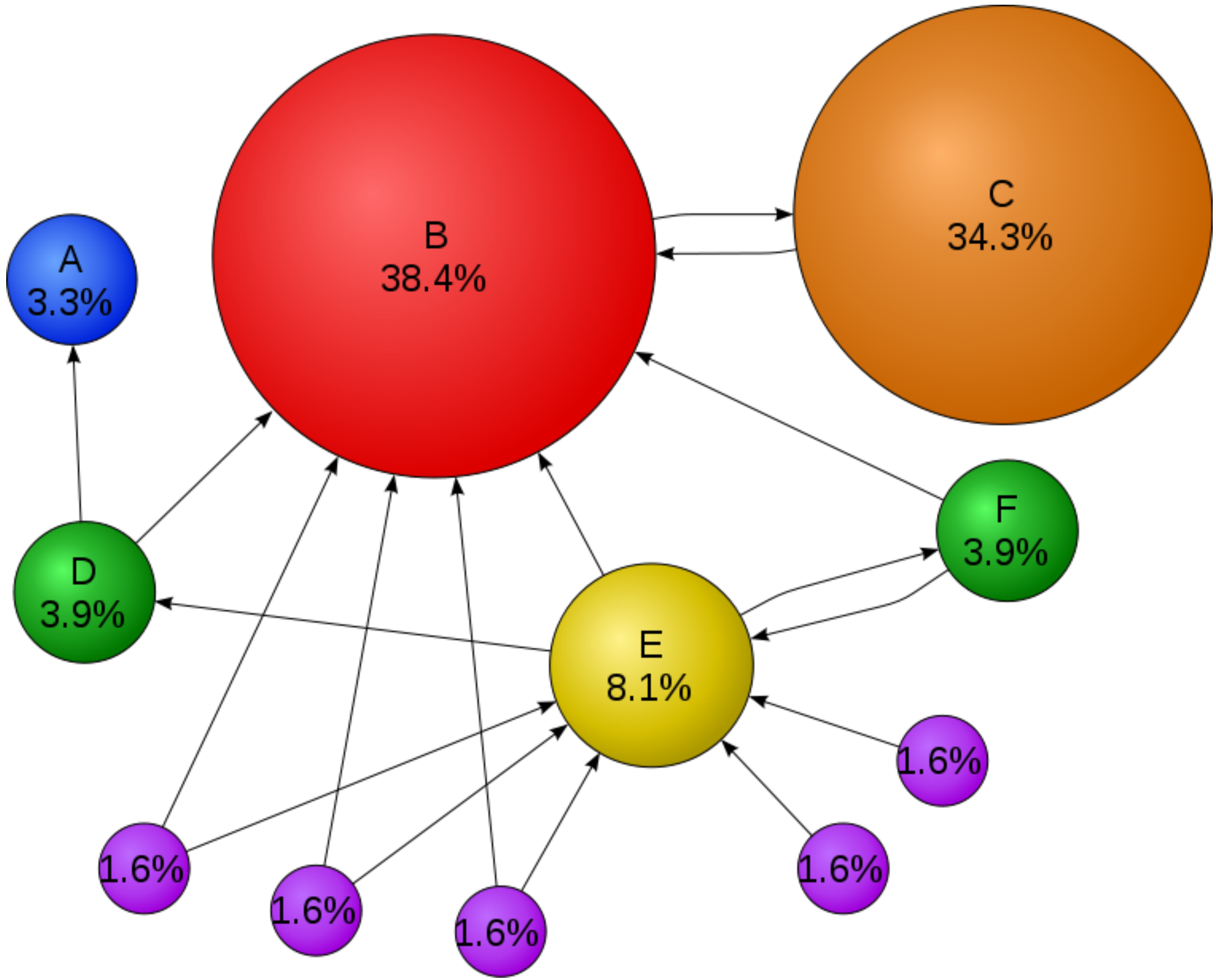
probability of being in state **A** in the long run

PageRank of webpage **A**

=

The stationary probability of **A**

Google PageRank



Google PageRank

Google:

"PageRank continues to be the heart of our software."

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