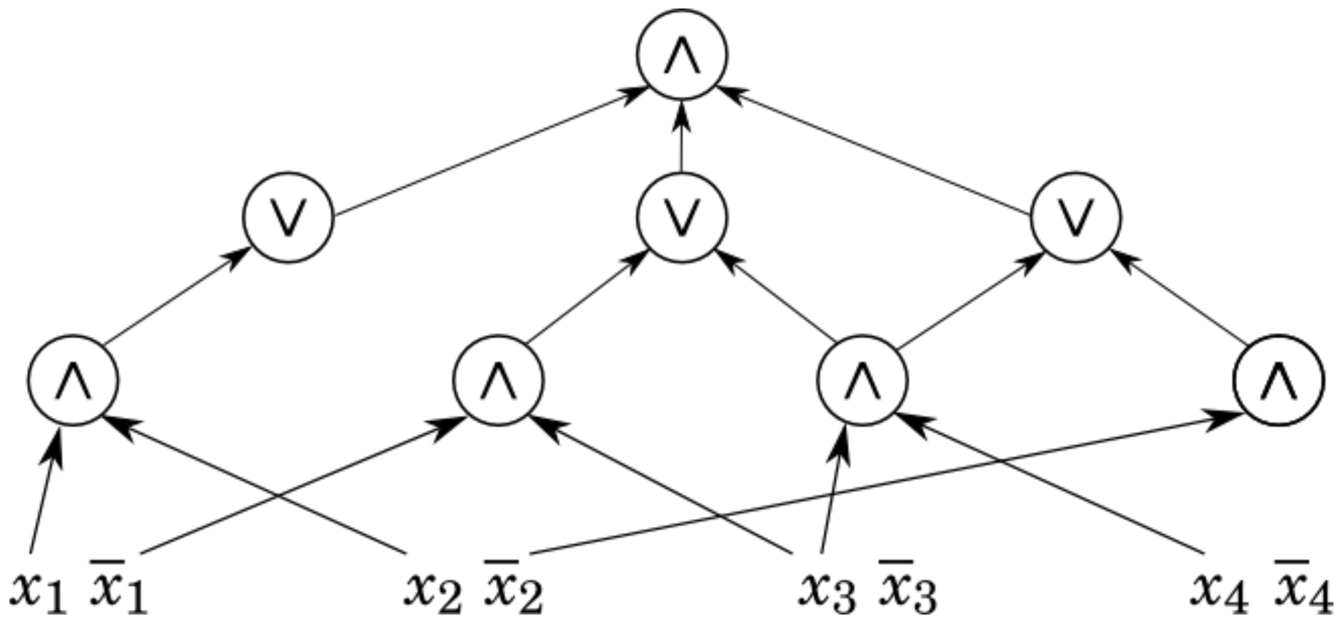
15-251
Great Theoretical Ideas in Computer Science

Lecture 9:
Boolean Circuits

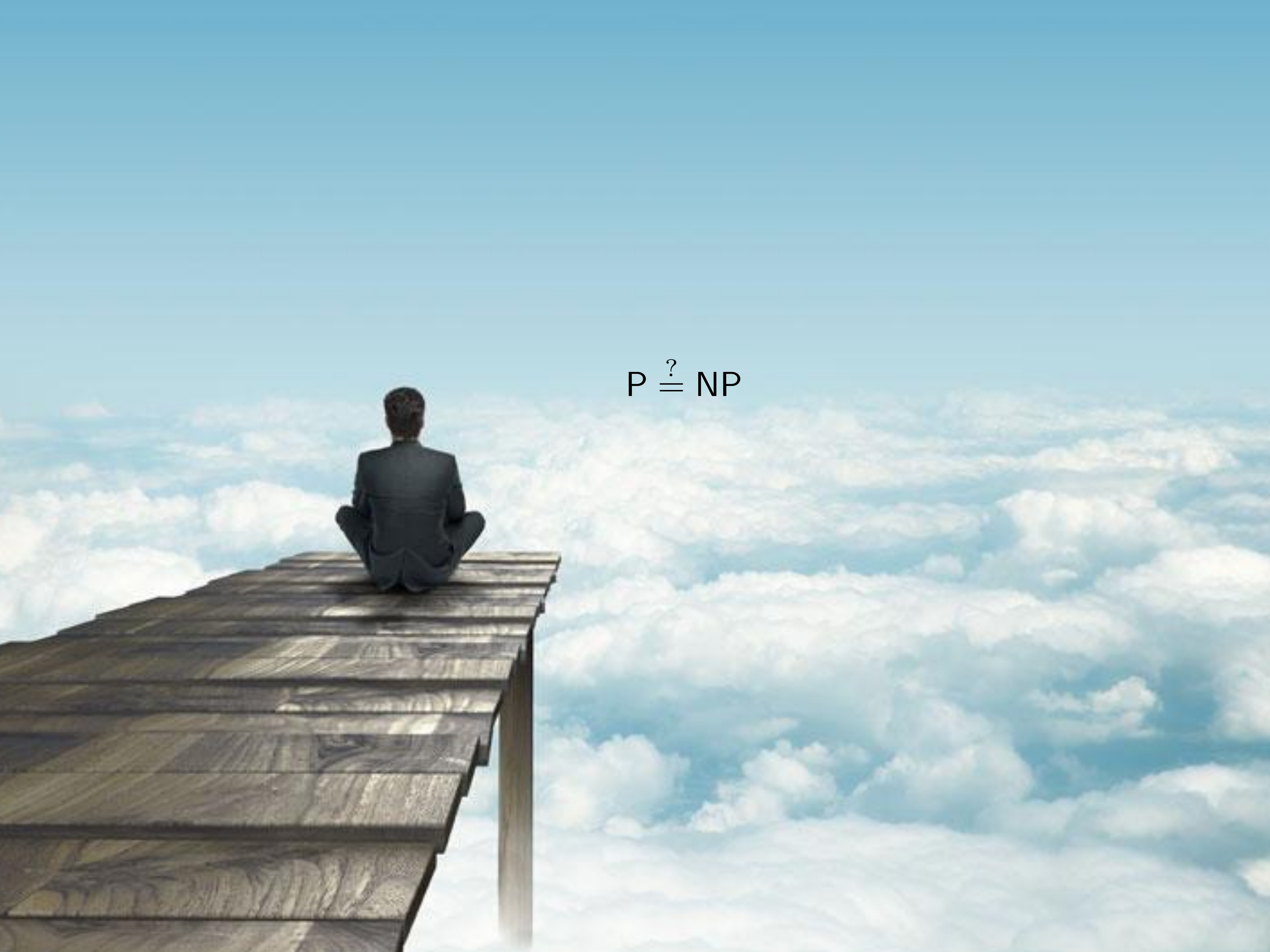


September 29th, 2015

Where we are, where we are going

Computer science is no more about computers than astronomy is about telescopes.

Monday	Tuesday	Wednesday	Thursday	Friday
Aug 31	Sep 1 ***Introduction***	Sep 2	Sep 3 ***On proofs***	Sep 4 *Quiz 1*
Sep 7	Sep 8 ***Finite automata***	Sep 9 hw1 w.s.	Sep 10 ***Turing machines***	Sep 11 *Quiz 2*
Sep 14	Sep 15 ***Uncountability***	Sep 16 hw2 w.s.	Sep 17 ***Undecidability***	Sep 18 *Quiz 3*
Sep 21	Sep 22 ***Intro to complexity 1***	Sep 23 hw3 w.s.	Sep 24 ***Intro to complexity 2***	Sep 25 *Quiz 4*
Sep 28	Sep 29 ***Circuit complexity***	Sep 30 hw4 w.s.	Oct 1 ***Graphs 1***	Oct 2 *Quiz 5*
Oct 5	Oct 6 ***Graphs 2***	Oct 7 hw5 w.s.	Oct 8 ***Graphs 3***	Oct 9 *Quiz 6*
Oct 12	Oct 13 ***Reductions***	Oct 14 Midterm 1	Oct 15 ***NP-completeness***	Oct 16 *Quiz 7*



$$P \stackrel{?}{=} NP$$

P vs NP is on the horizon

CMI

ABOUT PROGRAMS MILLENNIUM PROBLEMS PEOPLE PUBLICATIONS EUCLID EVENTS

Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

I million dollar question

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

(or maybe 6 million dollar question)

P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

P = NP ???

Navier–Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

Poincaré Conjecture

In 1904 the French mathematician Henri Poincaré asked if the three dimensional sphere is characterized as the unique simply connected three manifold. This question, the Poincaré conjecture, was a special case of Thurston's geometrization conjecture. Perelman's proof tells us that every three manifold is built from a set of standard pieces, each with one of eight well-understood geometries.

Birch and Swinnerton-Dyer Conjecture

Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod p to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of numbers into primes, and cryptography, to name three.

Computational complexity of an algorithm

Recall:

Definition:

The **running time** of an algorithm *A* is defined as

worst-case

$$T_A(n) = \max_{\substack{\text{instances } I \\ \text{of size } n}} \{\# \text{ steps } A \text{ takes on } I\}$$

Computational complexity of a **problem**

The intrinsic complexity of a problem:

Complexity of the best algorithm computing the problem.

How to show an upper bound on the intrinsic complexity?

> Give an algorithm that solves the problem.

How to show a lower bound on the intrinsic complexity?

> Argue against **all** possible algorithms that solve the problem.

The dream: Get a matching upper and lower bound.

What is P ?

P

The set of languages that can be decided in $O(n^k)$ steps for some constant k.

The theoretical divide between efficient and inefficient:

$L \in$ P ⟶ efficiently solvable.

$L \notin$ P ⟶ not efficiently solvable.

What is P ?

In practice:

$O(n)$	Awesome! Like really awesome!
$O(n \log n)$	Great!
$O(n^2)$	Kind of efficient.
$O(n^3)$	Barely efficient. (???)
$O(n^5)$	Would not call it efficient.
$O(n^{10})$	Definitely **not** efficient!
$O(n^{100})$	WTF?

Why P ?

- P is not meant to mean "efficient in practice"
- It means "You have done something extraordinarily better than brute force (exhaustive) search."
- So P is about mathematical insight into a problem's structure.
- Robust to notion of what is an elementary step, what model we use, reasonable encoding of input, implementation details.
- Wouldn't make sense to cut it off at some specific exponent.
- Plus, big exponents don't really arise.
- If it does arise, usually can be brought down.

Why P ?

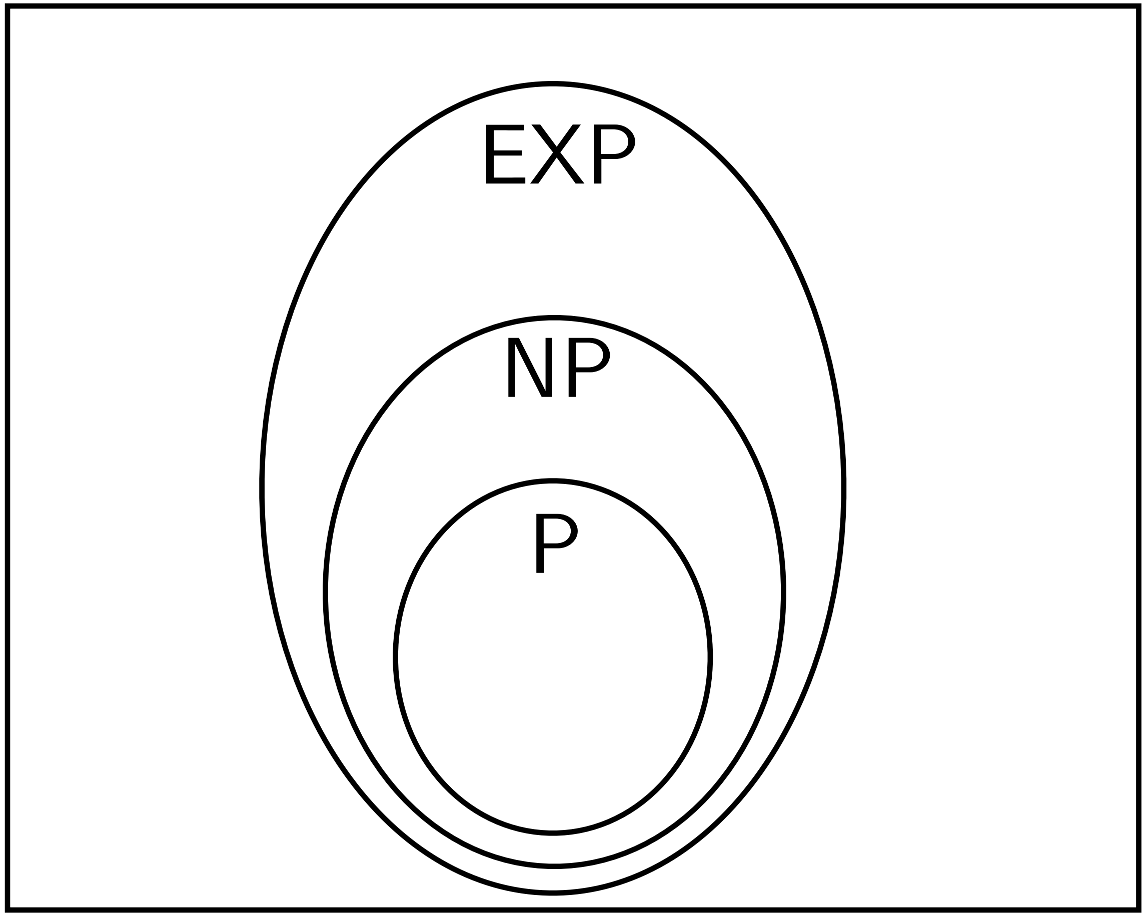
Summary: Being in P vs not being in P
is a qualitative difference, not a quantitative one.

What is NP ?

EXP

The set of languages that can be decided in $O(k^n)$ steps for some constant $k > 1$.

DECIDABLE LANGUAGES

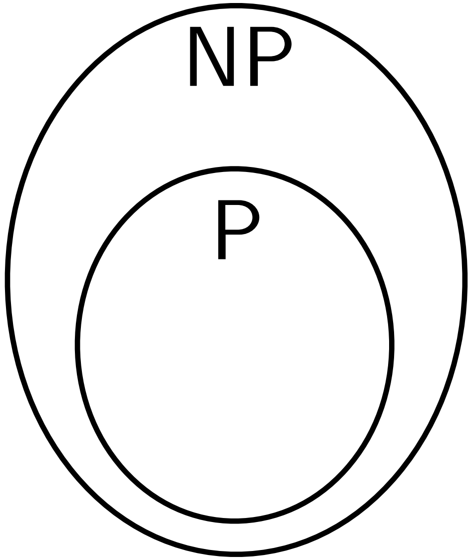


NP :

A class between

P and EXP.

What is NP ?



$$P \stackrel{?}{=} NP$$

asks whether these two sets are equal.

How would you show $P = NP$?

Show that every problem in NP can be solved in poly-time.

How would you show $P \neq NP$?

Show that there is a problem in NP which cannot be solved in poly-time.

You have to argue against all possible poly-time TMs.

Boolean Circuits

Some preliminary questions

What is a Boolean circuit?

- It is a computational model for computing decision problems (or computational problems).

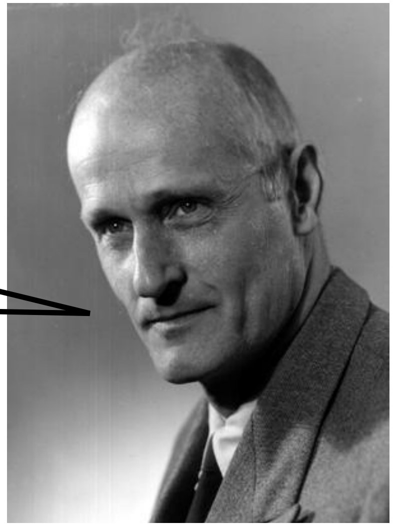
We already have TMs. Why Boolean circuits?

- The definition is simpler.
- Easier to understand, usually easier to reason about.
- Boolean circuits can efficiently simulate TMs.
  (efficient decider TM $\Longrightarrow$ efficient/small circuits.)
- Circuits are good models to study *parallel computation*.
- Real computers are built with digital circuits.

Sounds awesome!
So why didn't we just learn about circuits first?

There is a small catch.

An algorithm is a finite answer to infinite number of questions.



Stephen Kleene
(1909 - 1994)

Sounds awesome!
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There is a small catch.

Circuits are an infinite answer to infinite number of questions.

Anil Ada
(???? - 2077)

Dividing a problem according to length of input

$$\Sigma = \{0, 1\}$$

$$L \subseteq \{0, 1\}^*$$

$$f : \{0, 1\}^* \to \{0, 1\}$$

$\{0, 1\}^n$ = all strings of length *n*

$$L_n = \{w \in L : |w| = n\}$$

$$f^n : \{0, 1\}^n \to \{0, 1\}$$

for $x \in \{0, 1\}^n$,

$$f^n(x) = f(x)$$

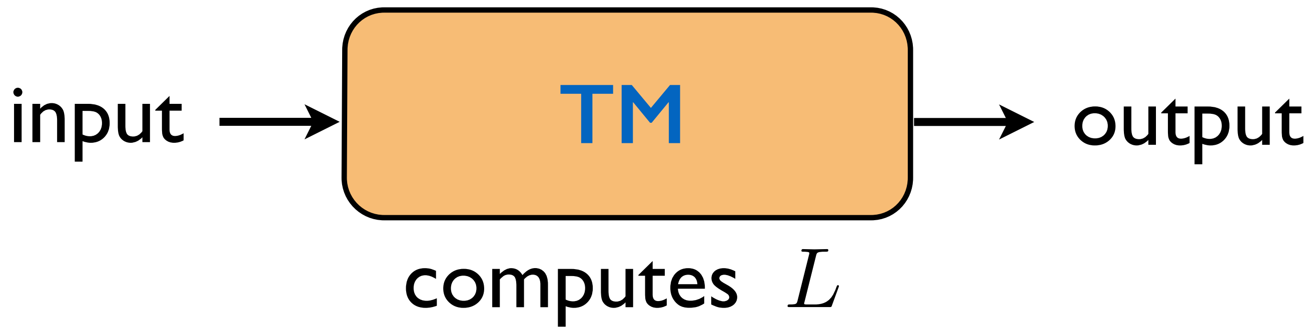
$$L = L_0 \cup L_1 \cup L_2 \cup \cdots$$

$$f = (f^0, f^1, f^2, \ldots)$$

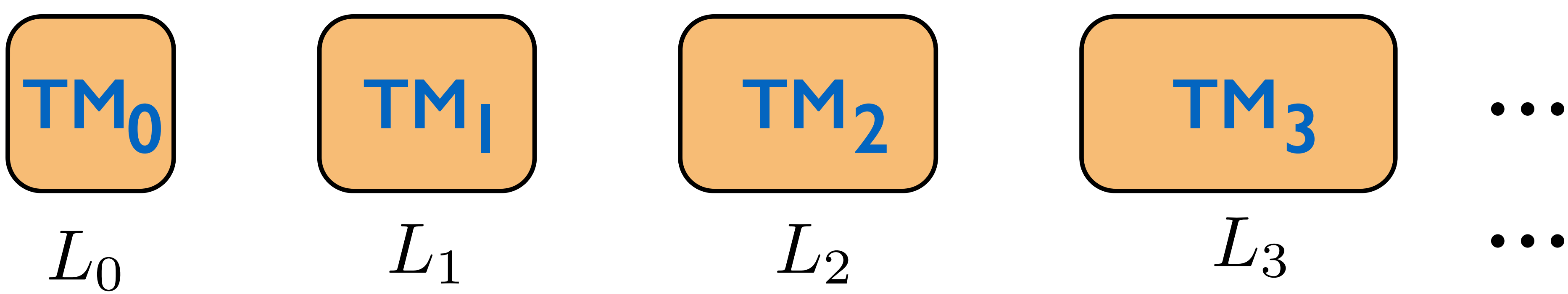
Dividing a problem according to length of input

A TM is a finite object (finite number of states)

but can handle any input length.



Imagine a model where we allow the TM to grow with input length.



Dividing a problem according to length of input

So one machine does not compute L.

You use a family of machines:

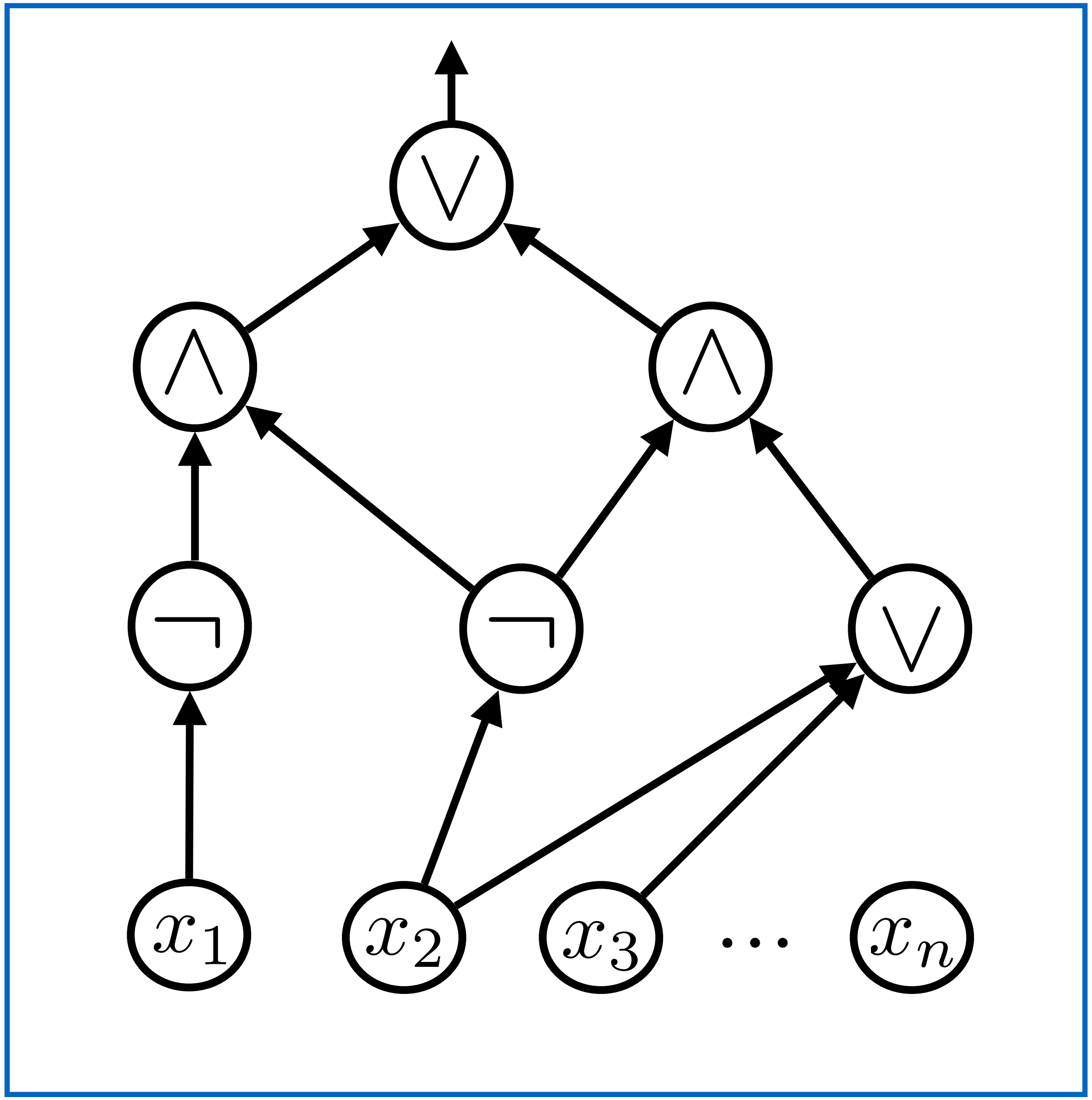
$$(M_0, M_1, M_2, \ldots)$$

(Imagine having a different Python function for each input length.)

Is this a reasonable/realistic model of computation?

Boolean circuits work this way.
Need a separate circuit for each input length.

Picture of a circuit



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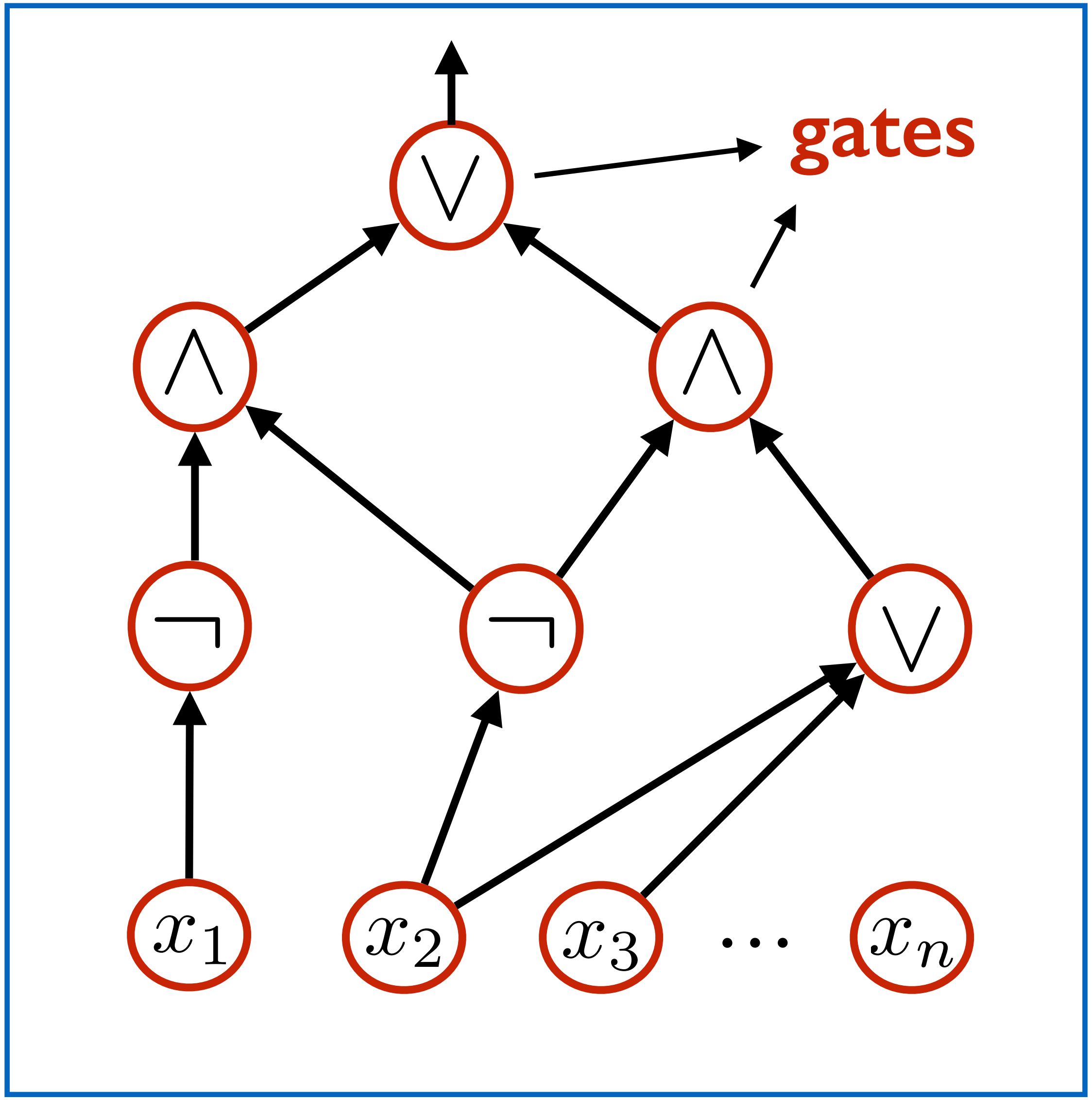

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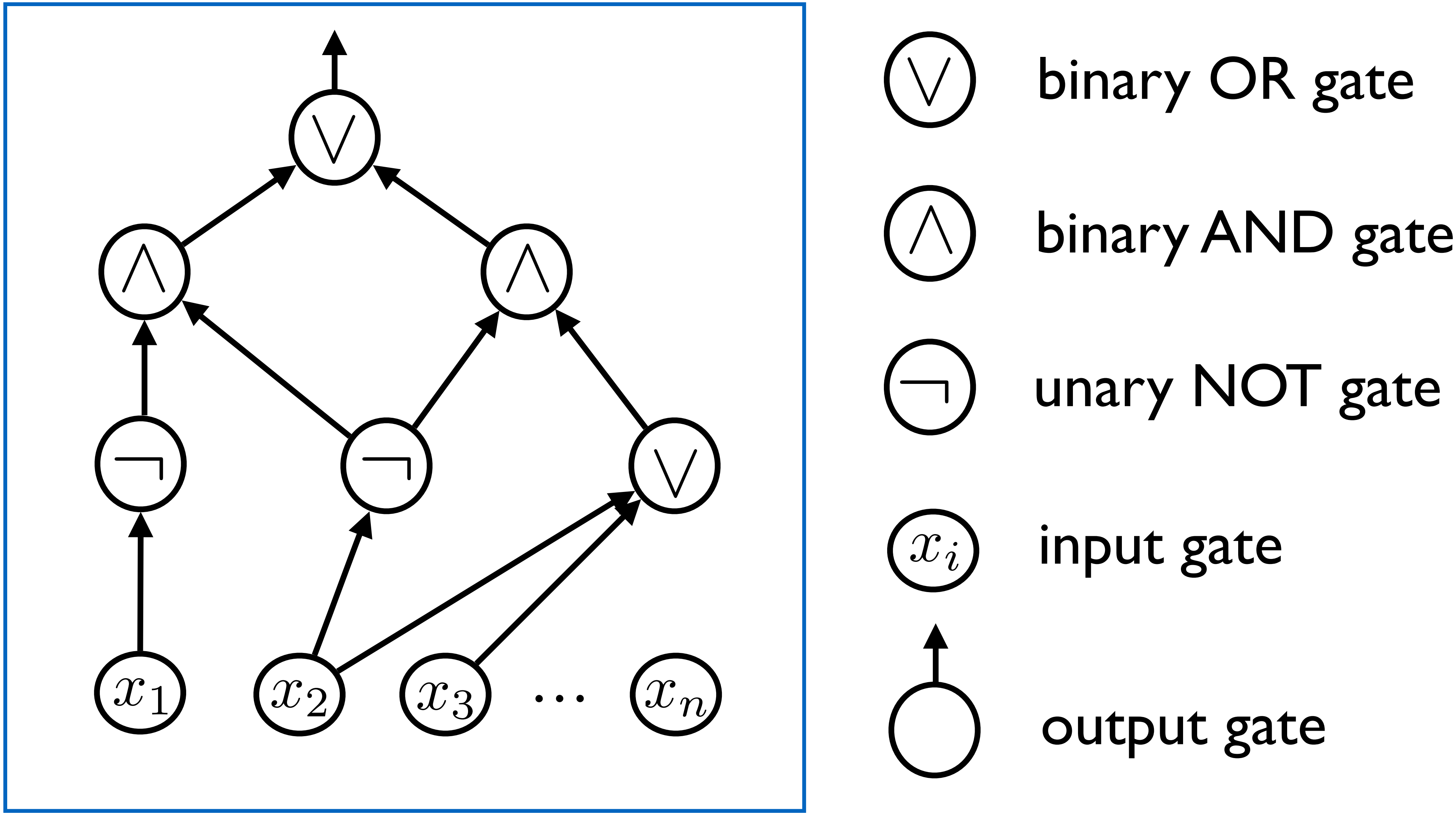

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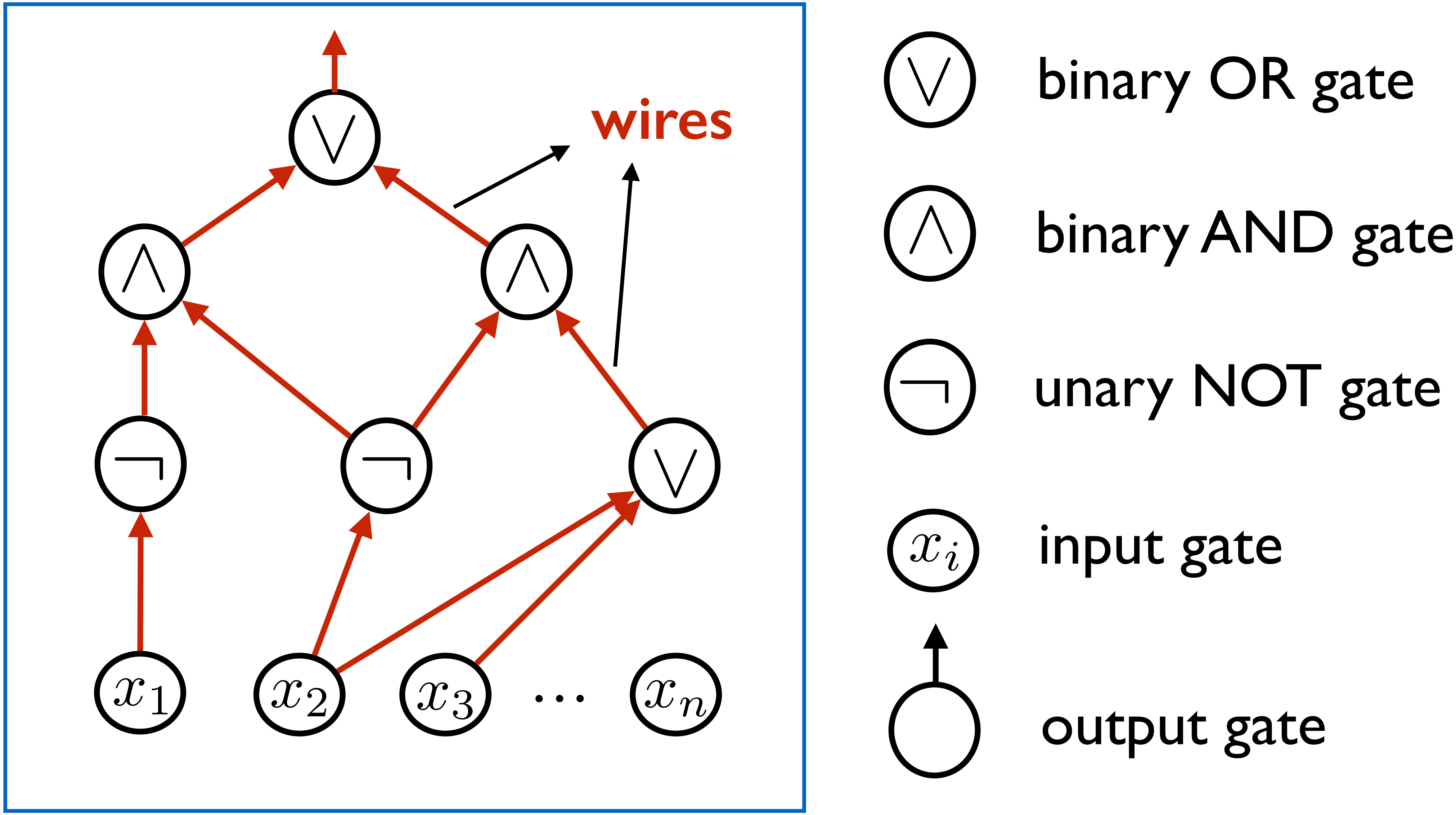

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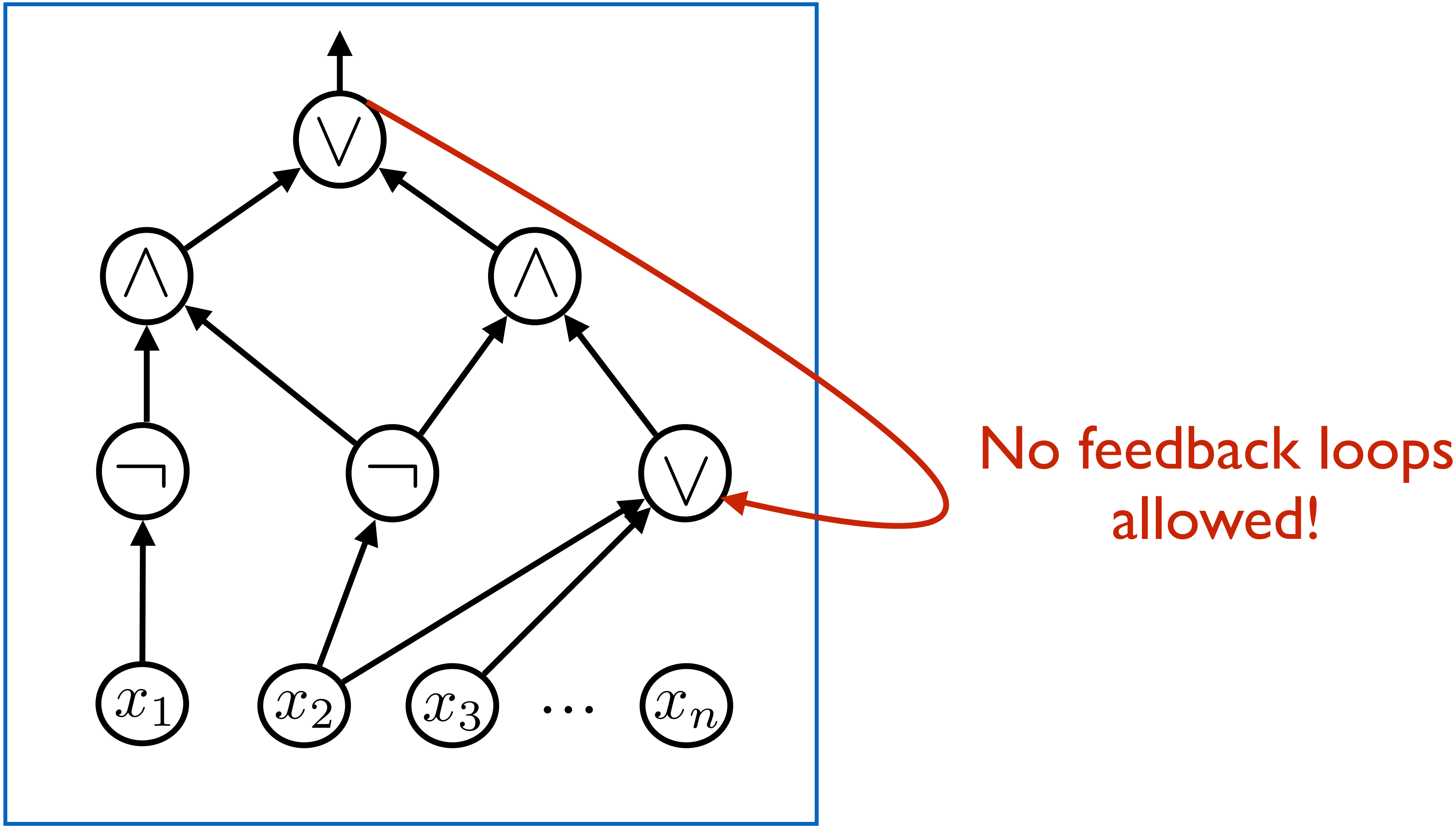
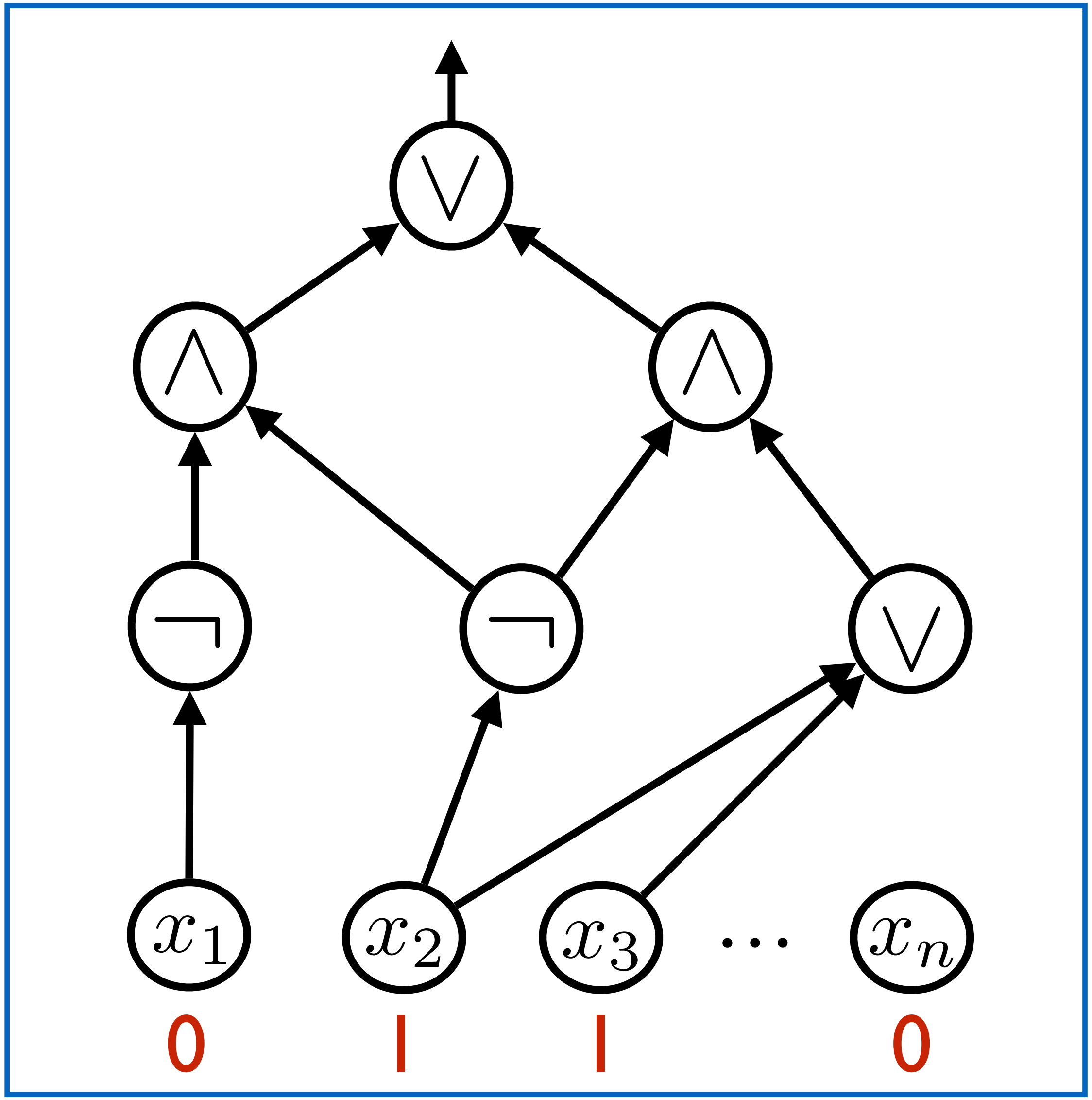
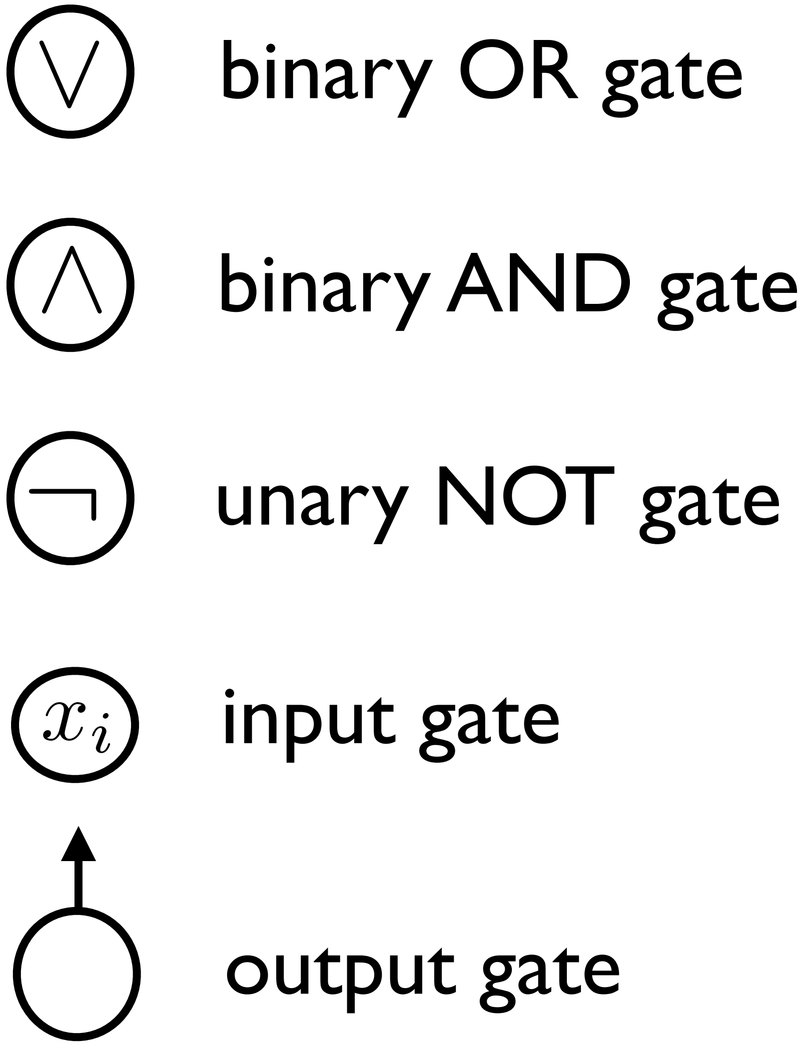


Information flows from input gates to the output gate.

Picture of a circuit

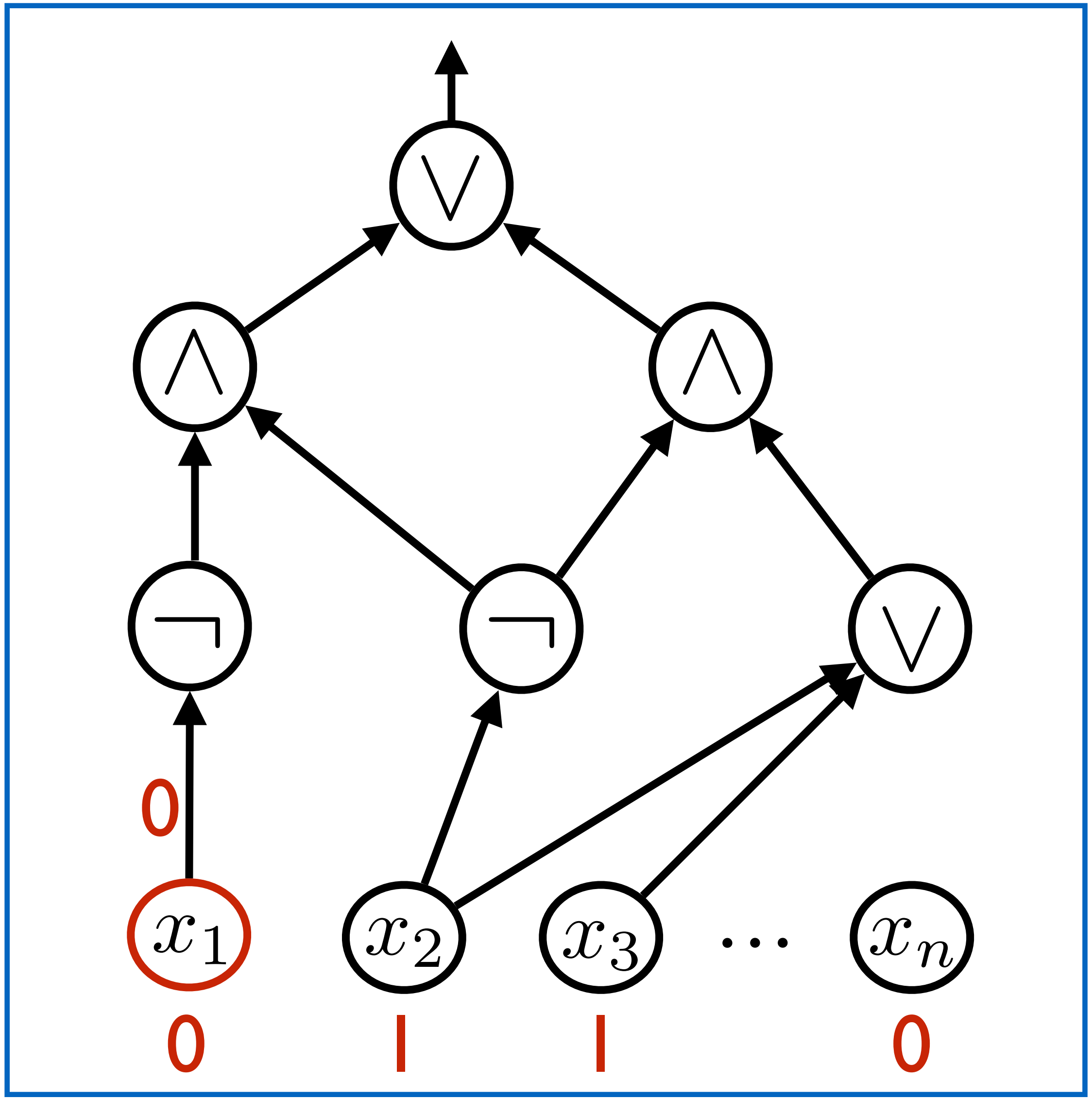


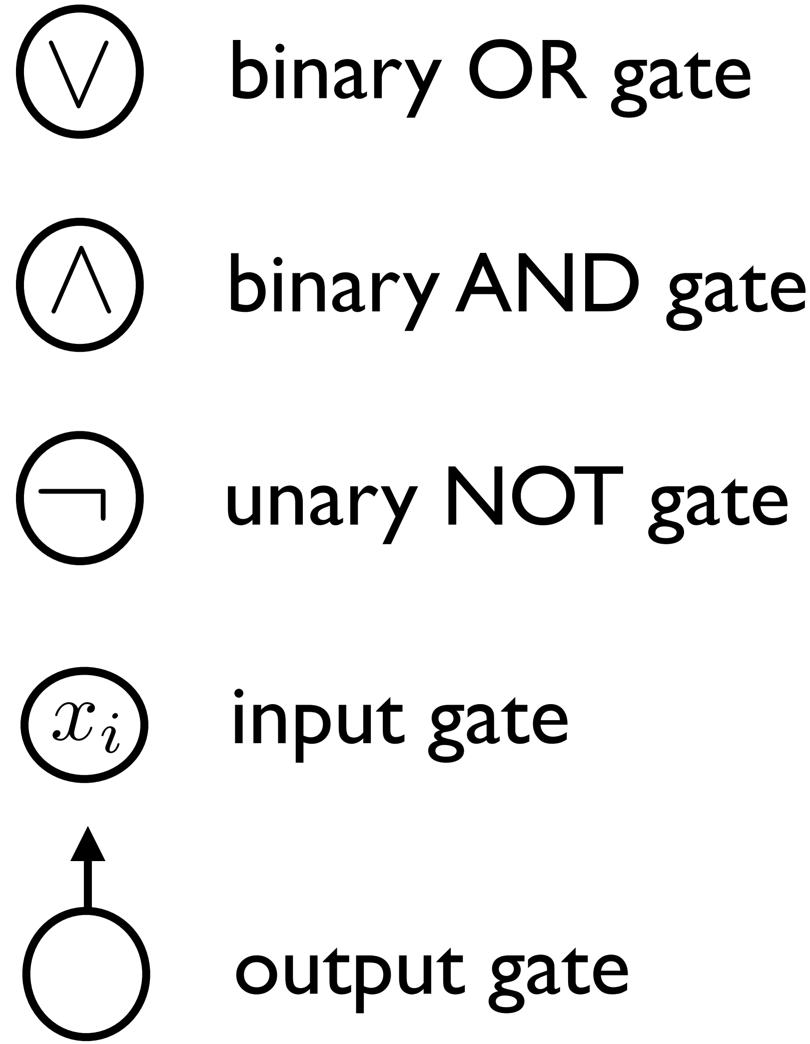


Computes a function $f : \{0,1\}^n \to \{0,1\}$.

So how does it compute $f(x_1, x_2, \ldots, x_n)$?

Picture of a circuit





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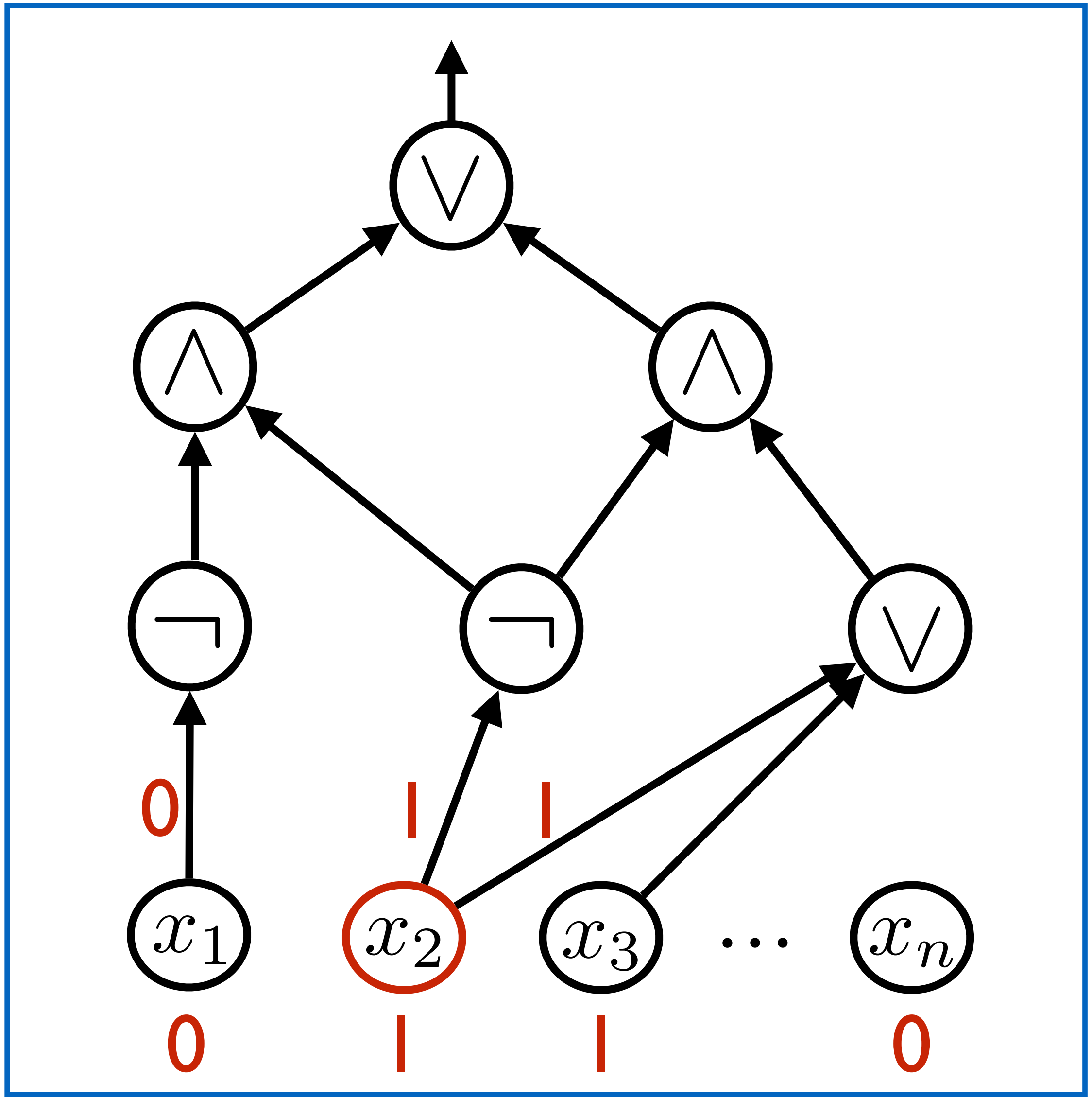

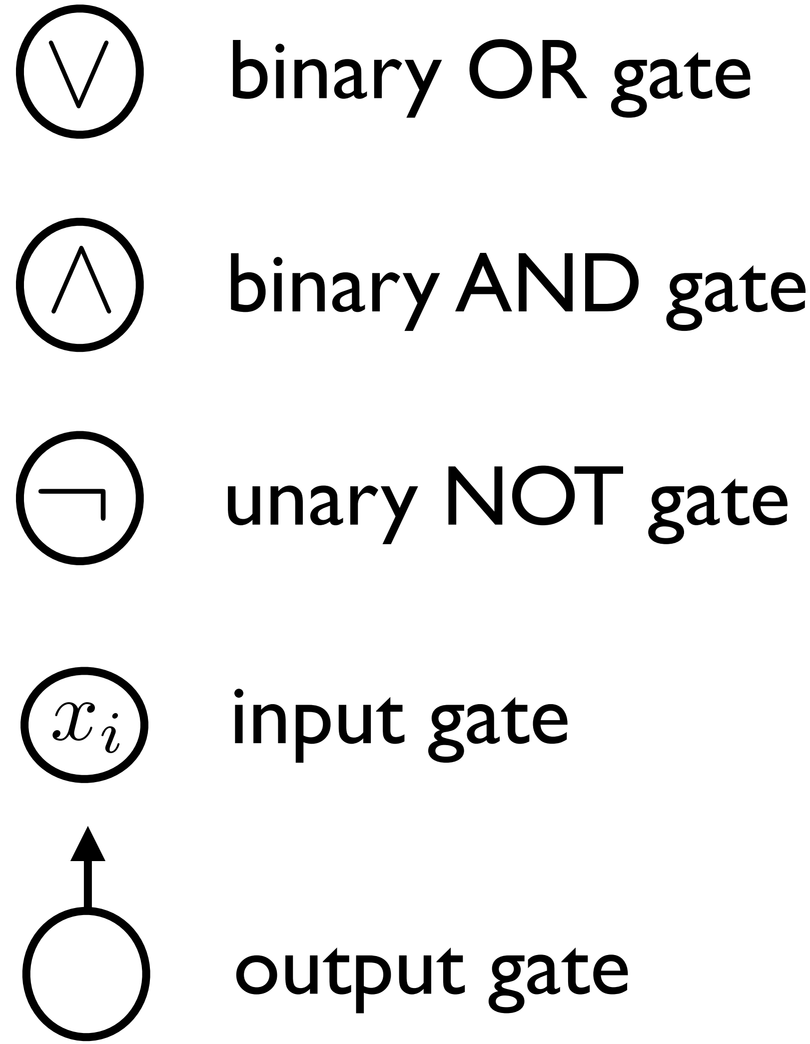

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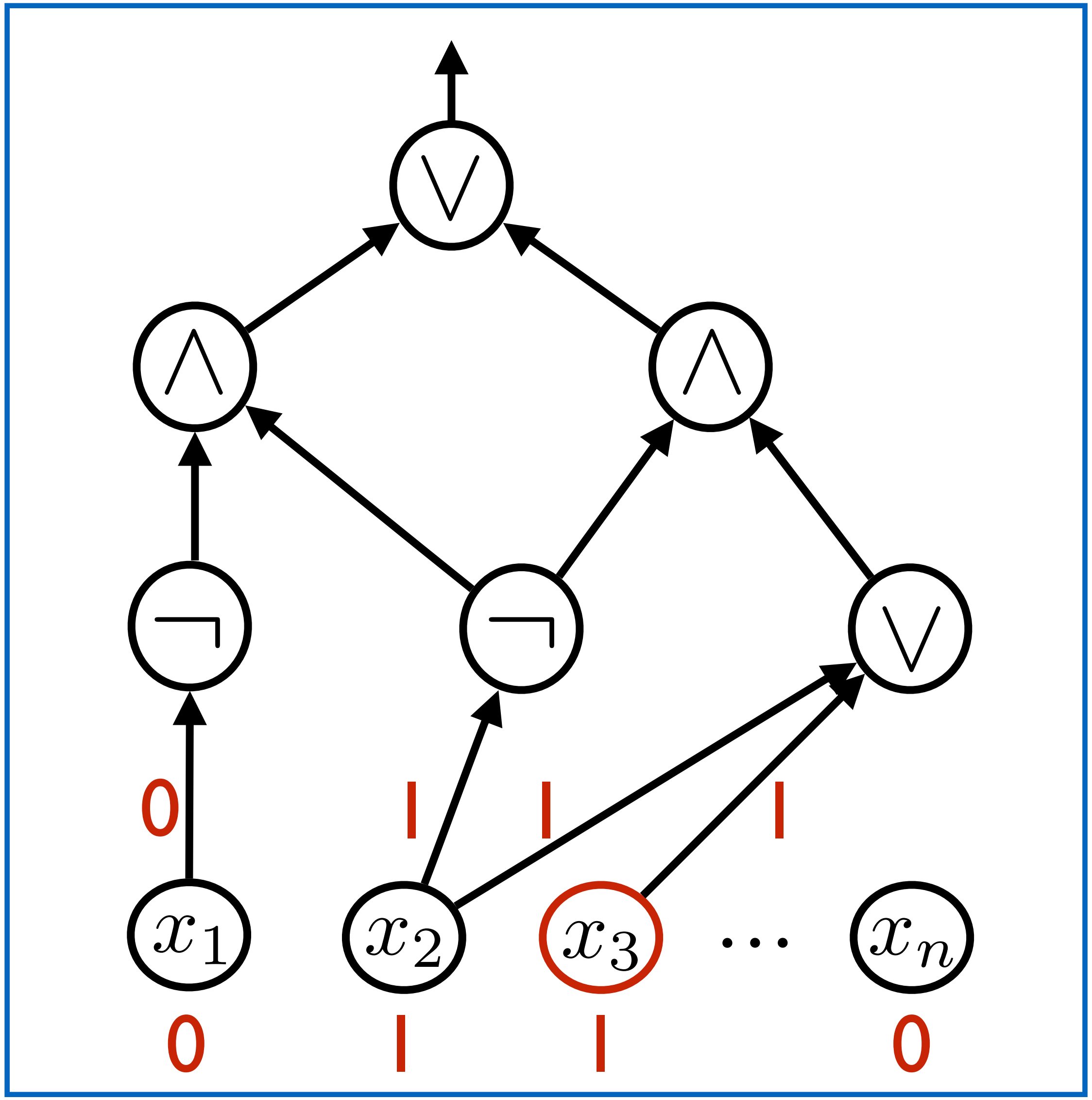

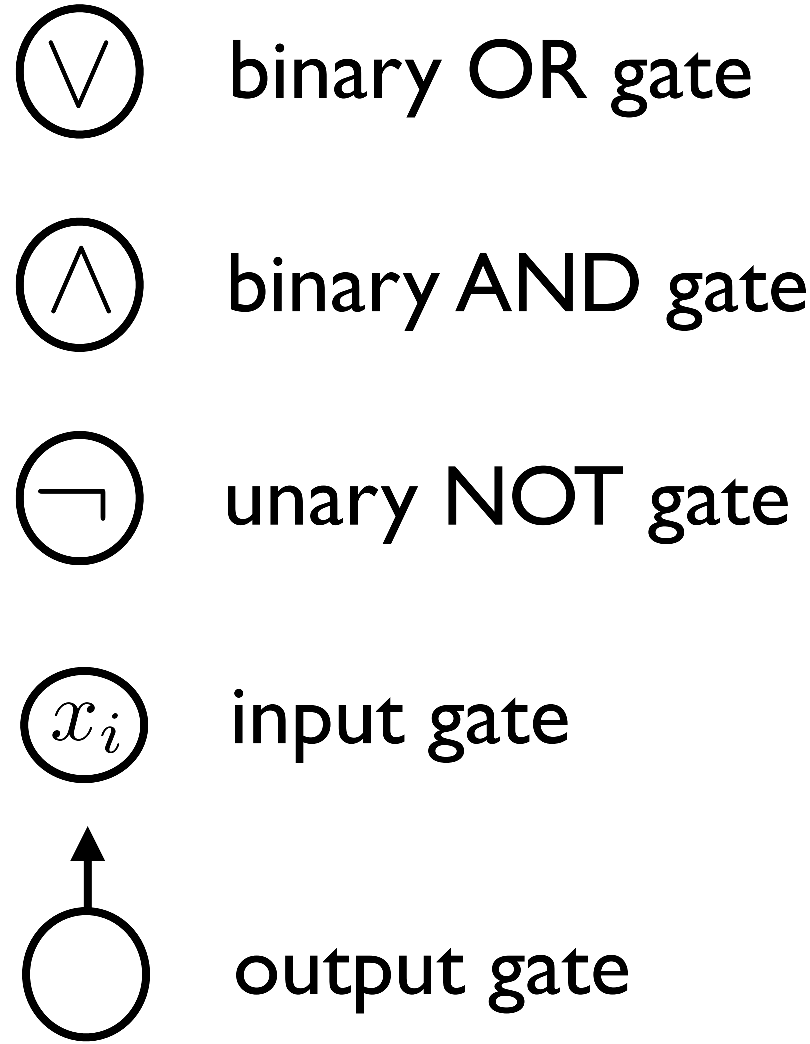

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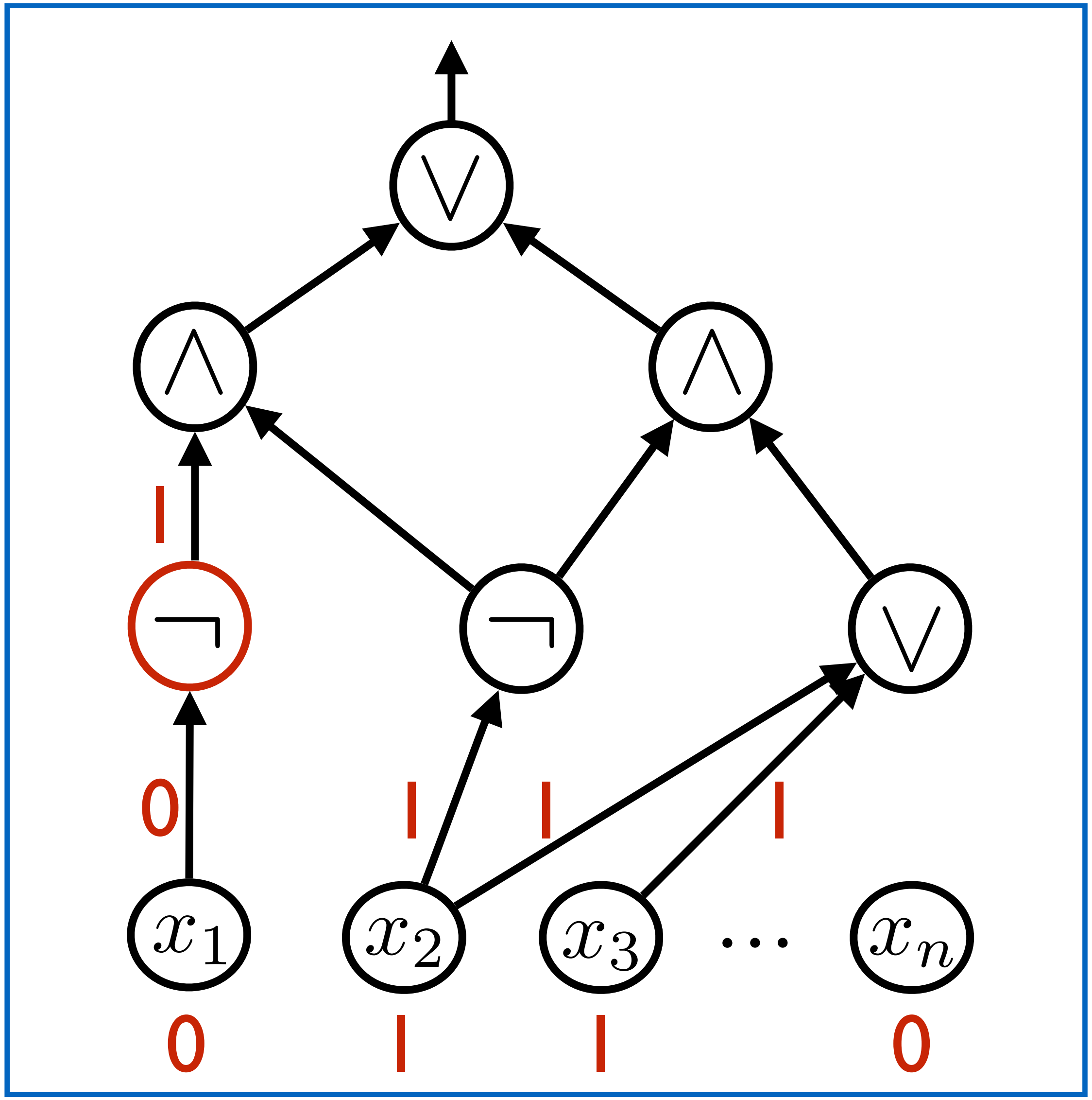

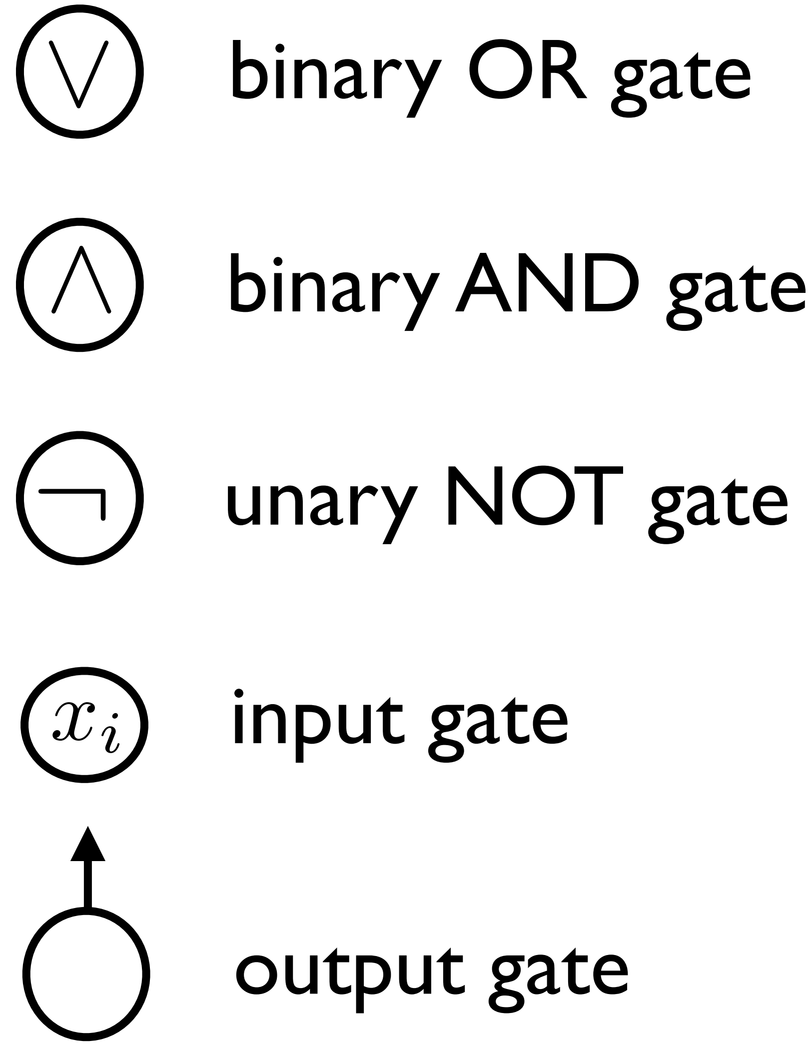

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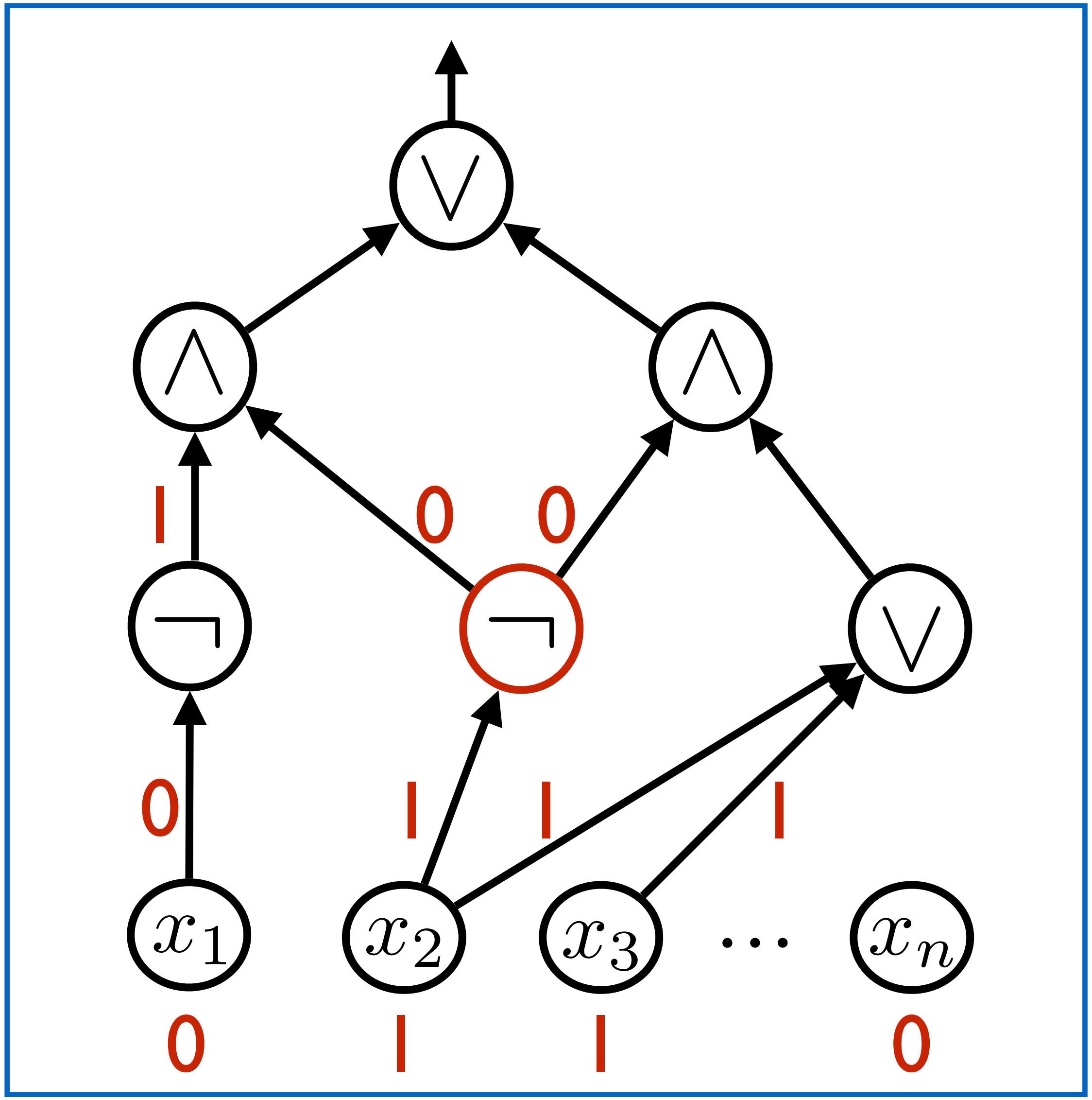

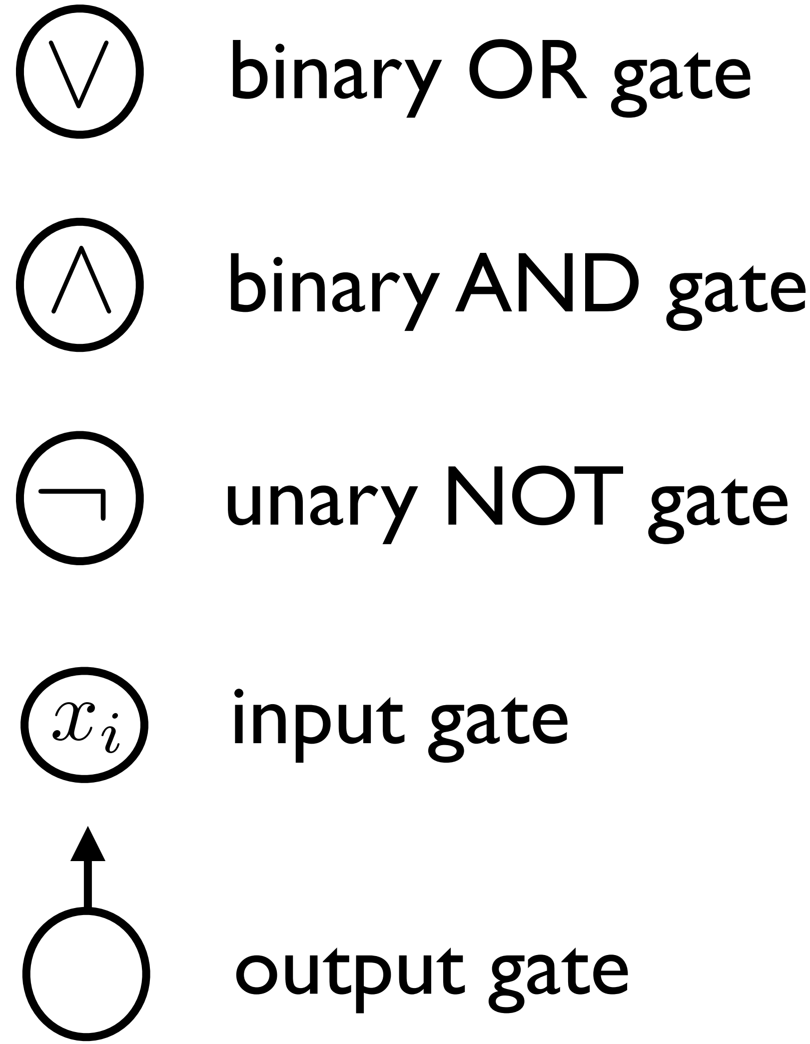

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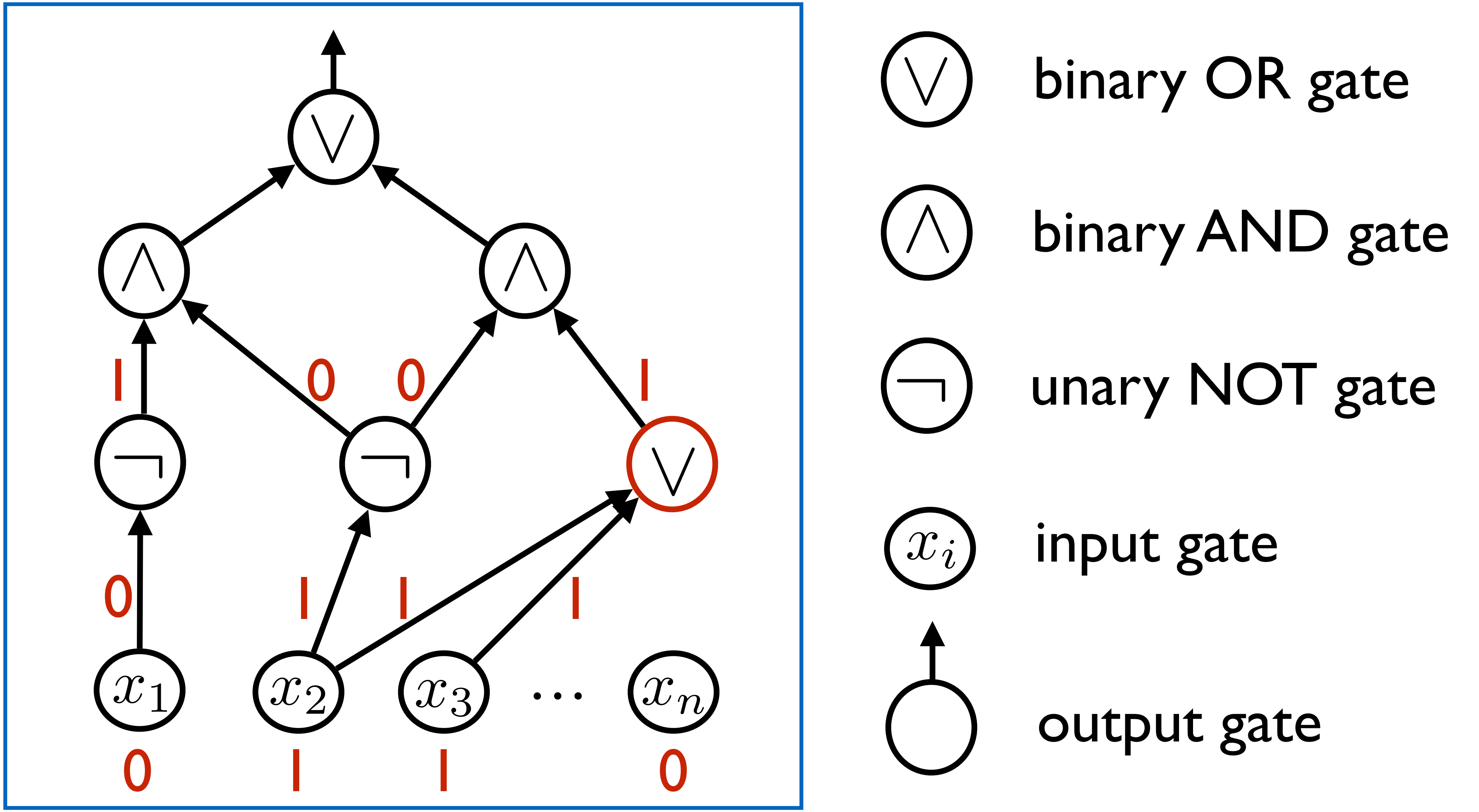

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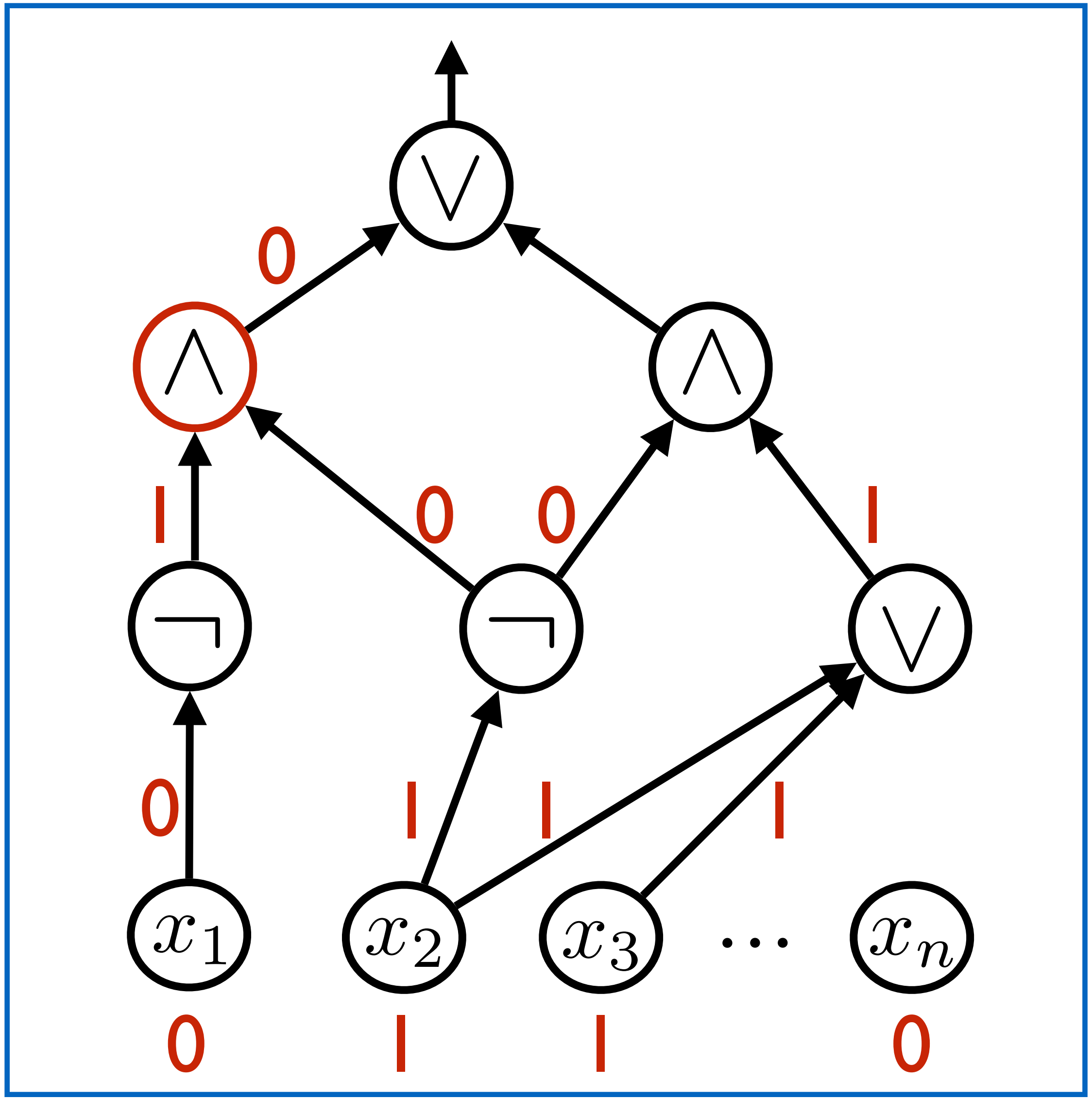

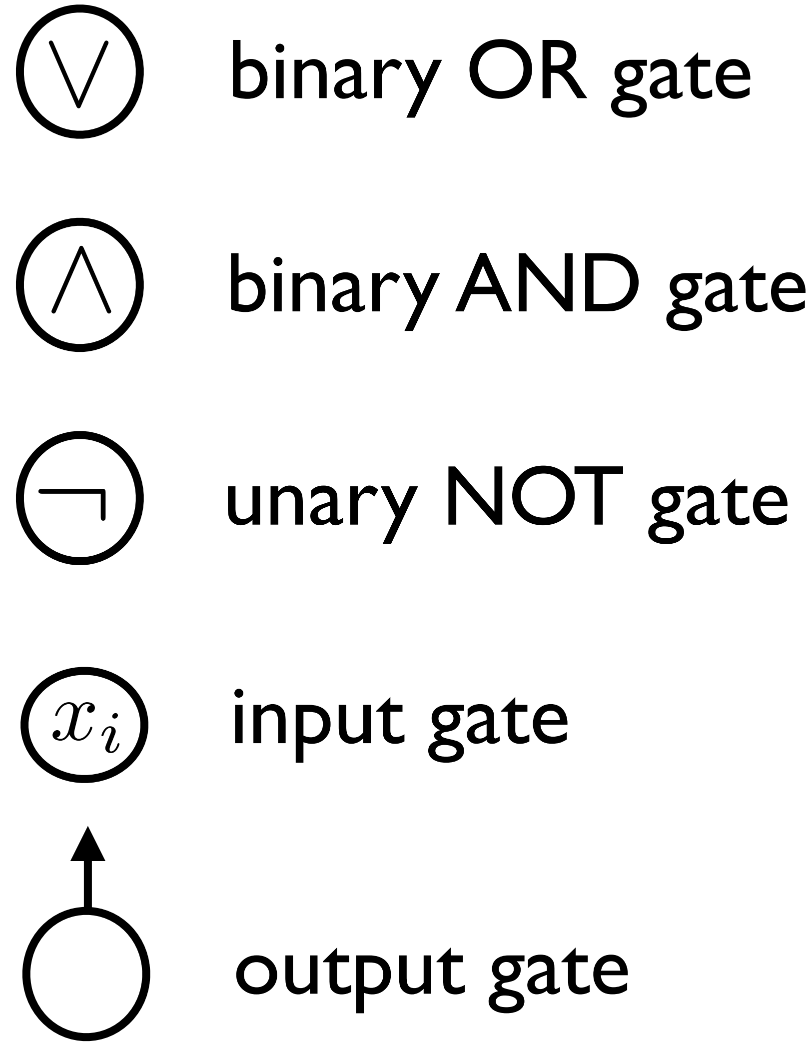

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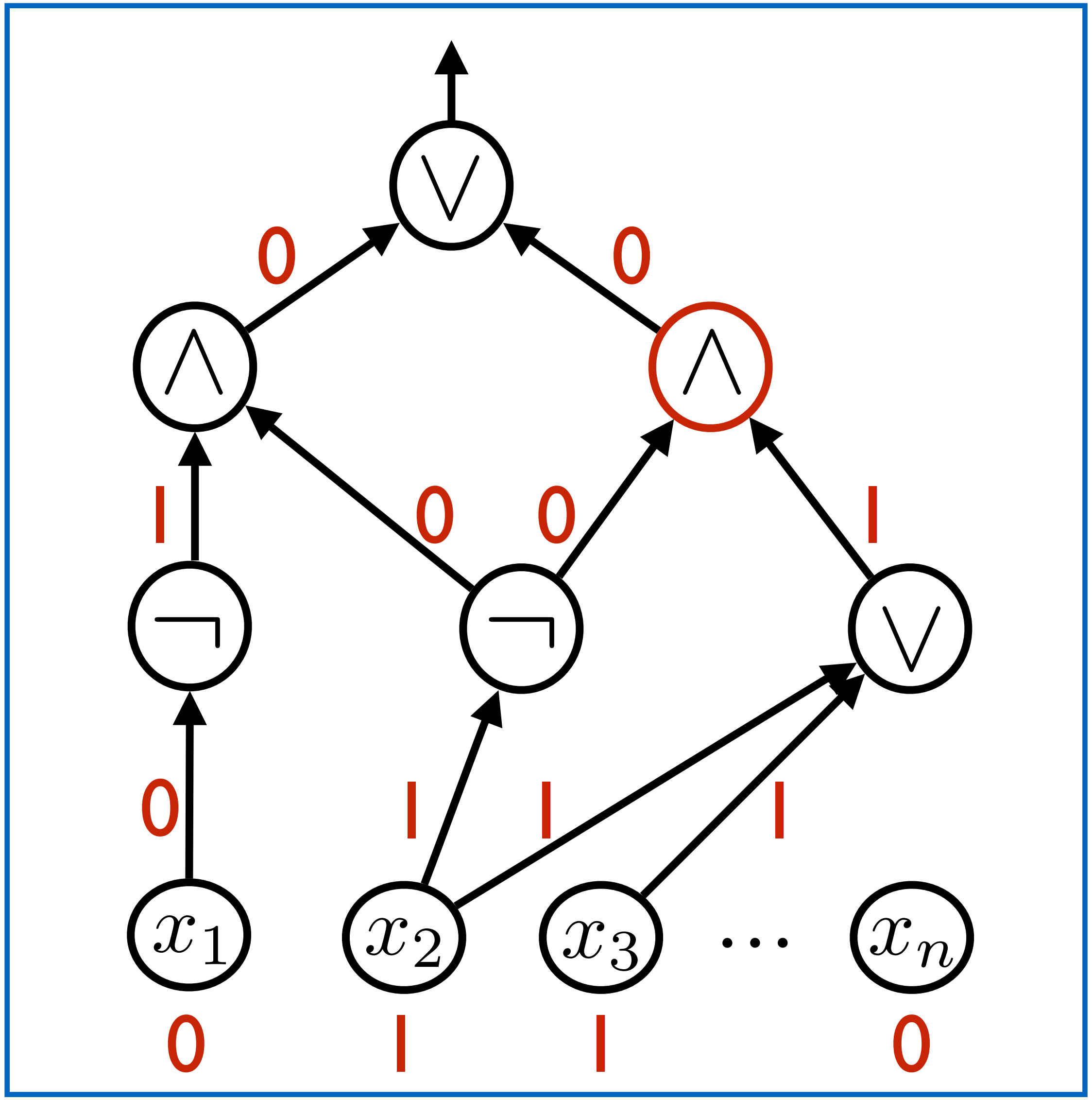

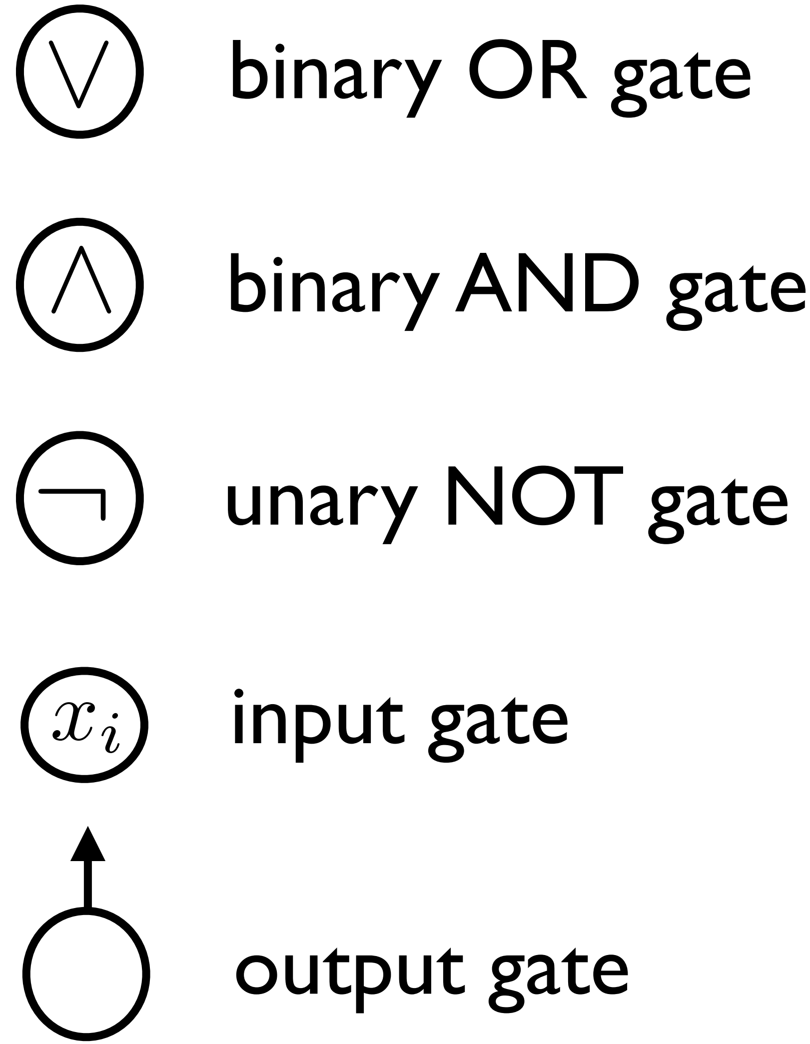

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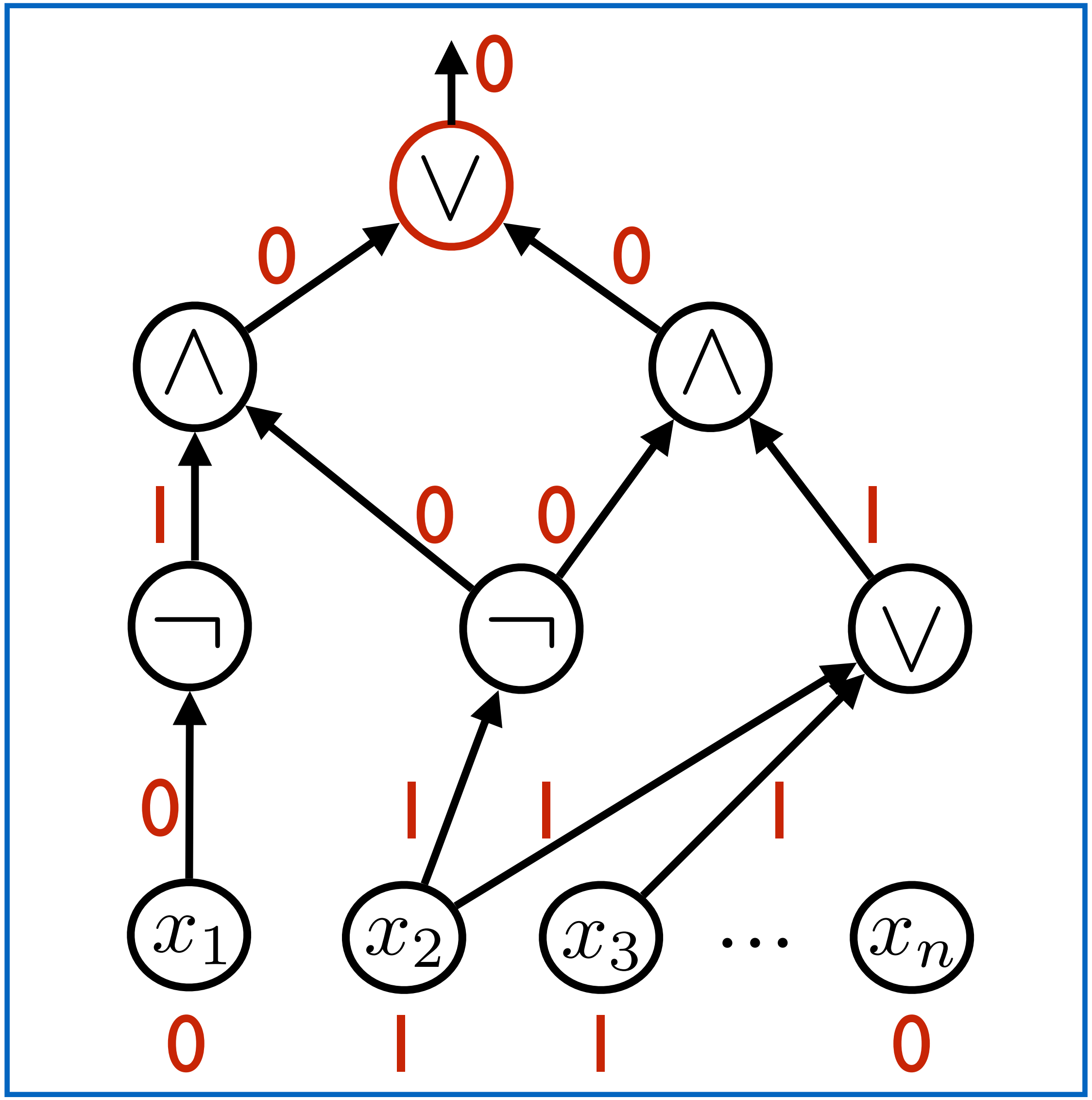

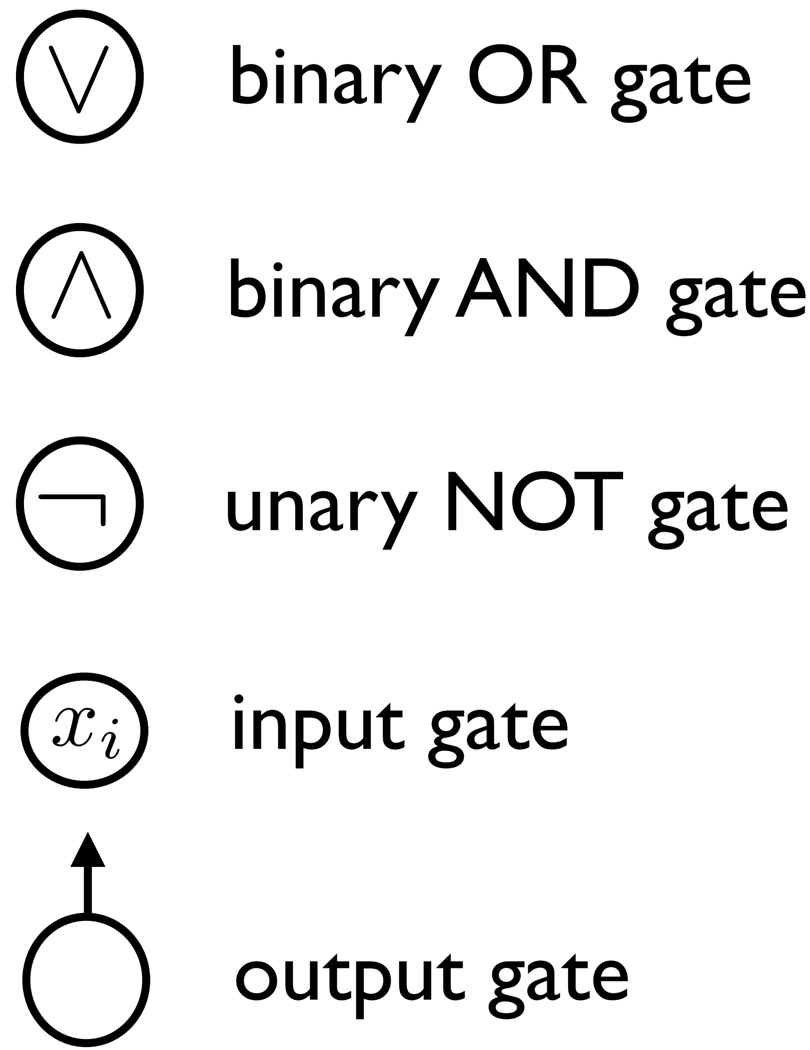

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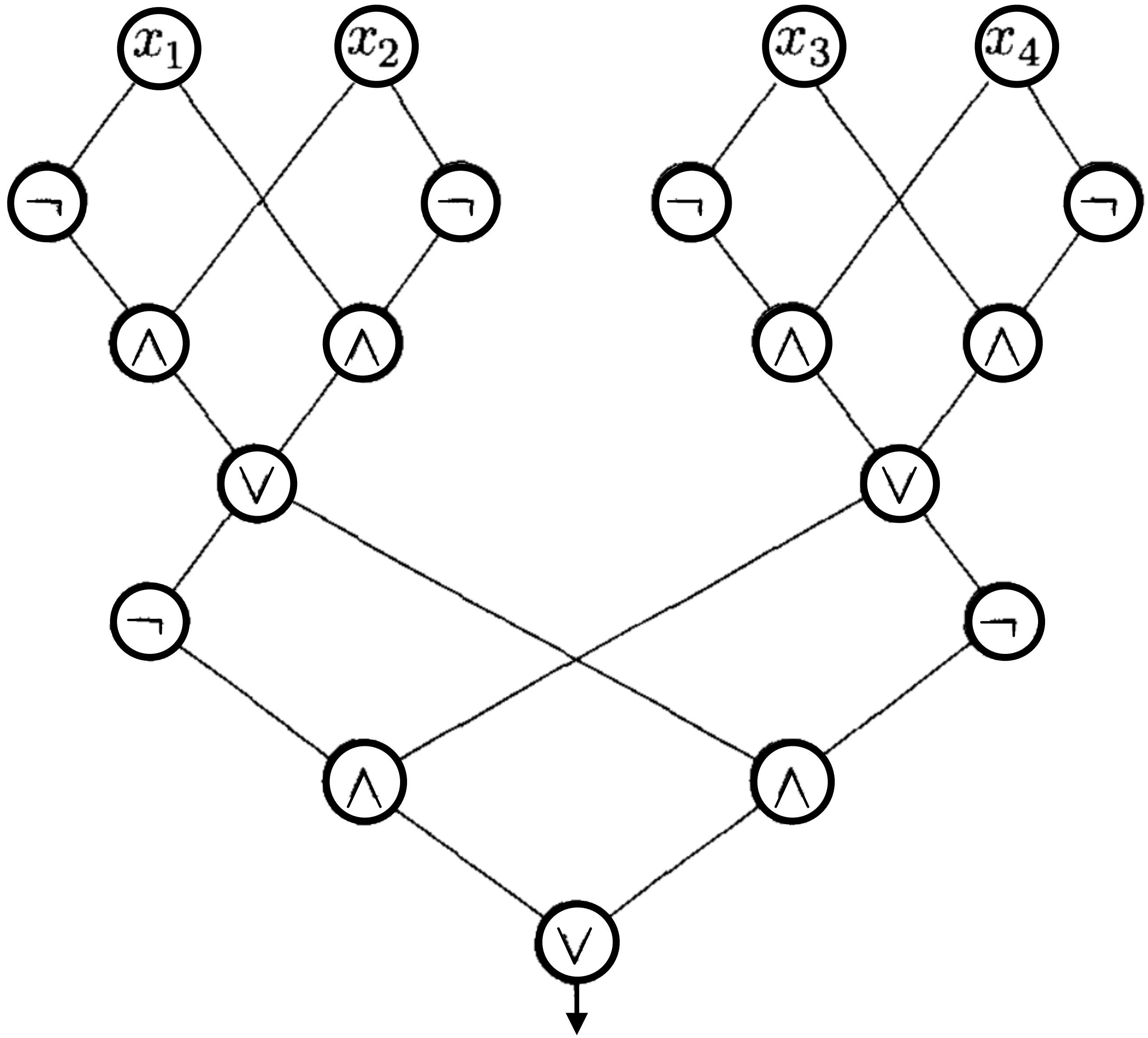
So how does it compute $f(x_1, x_2, \ldots, x_n)$?

Poll: What does this circuit compute ?

(sometimes circuits are drawn upside down)



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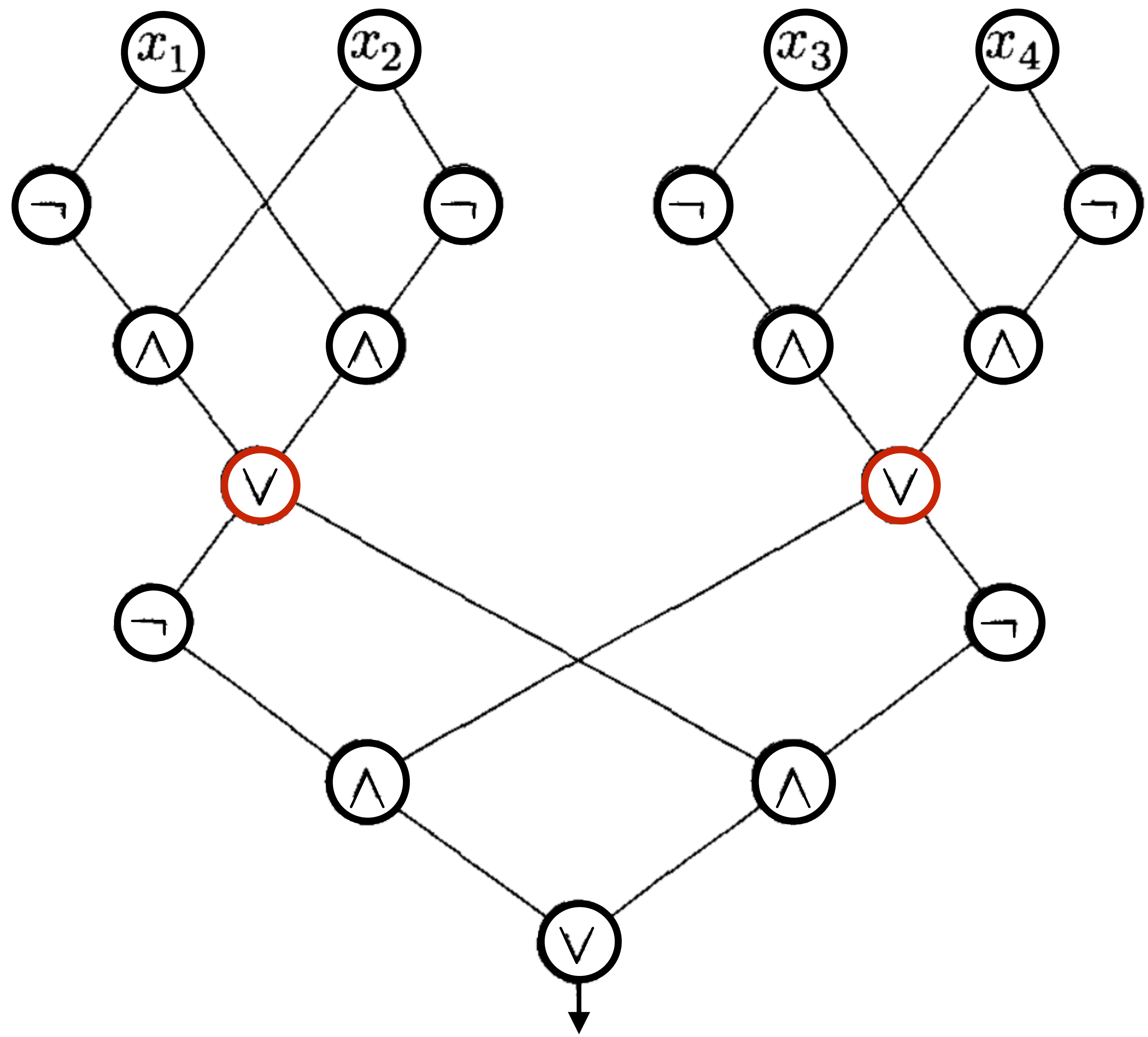

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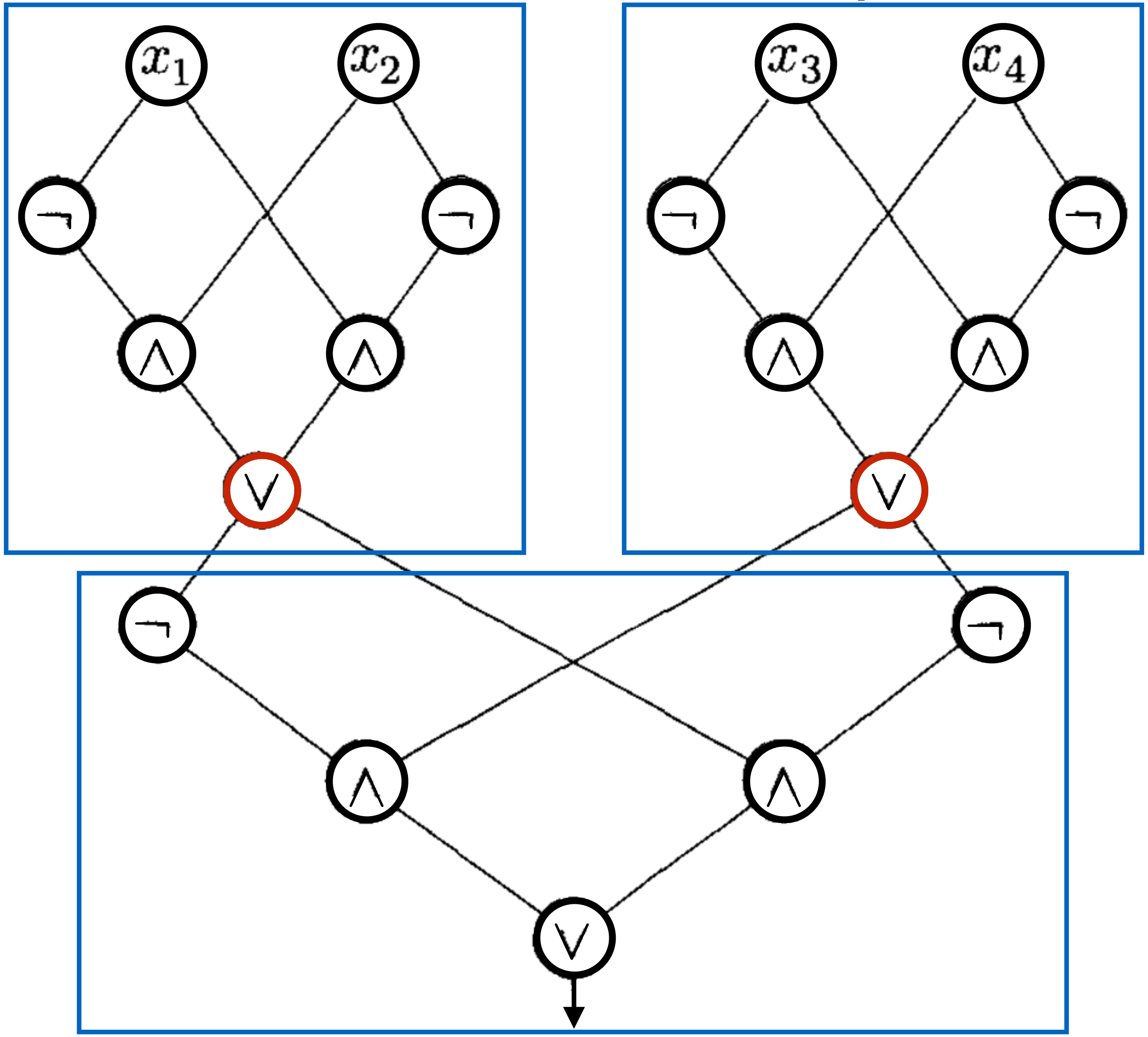
(sometimes circuits are drawn upside down)

parity of $x_1 + x_2$

$x_1 \oplus x_2$



parity of $x_3 + x_4$

$x_3 \oplus x_4$

$x_1 \oplus x_2 \oplus x_3 \oplus x_4$

How does a circuit decide/compute a language?

How do we measure the complexity of a circuit?

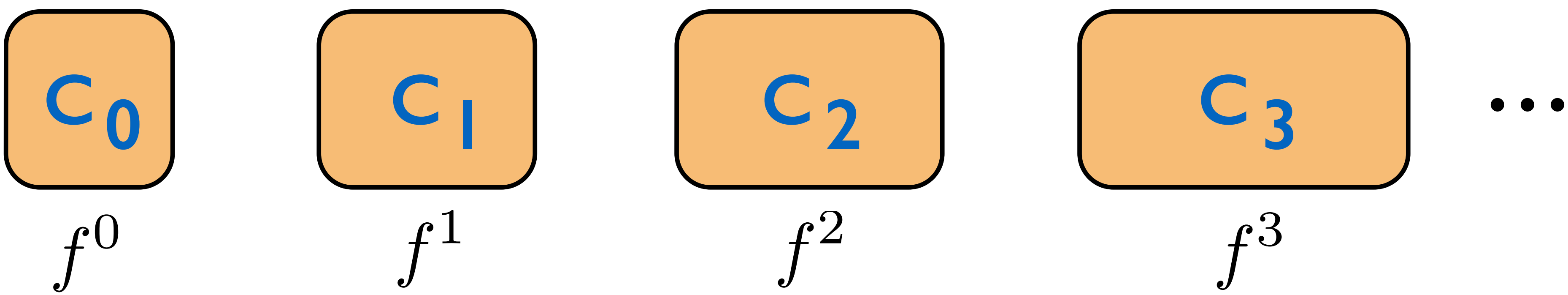
How can a circuit compute a language?

A circuit has a fixed number of inputs.

How can we compute/decide a decision problem $f : \{0,1\}^* \to \{0,1\}$ with circuits?

$$f = (f^0, f^1, f^2, \ldots) \quad \text{where} \quad f^n : \{0,1\}^n \to \{0,1\}$$

Construct a circuit for each input length.



A circuit family C is a collection of circuits $(C_0, C_1, C_2, \ldots)$ where each C_n takes n input variables.

How can a circuit compute a language?

A circuit has a fixed number of inputs.

How can we compute/decide a decision problem $f : \{0,1\}^* \to \{0,1\}$ with circuits?

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A circuit family C is a collection of circuits $(C_0, C_1, C_2, \ldots)$ where each C_n takes n input variables.

We say that a circuit family C decides/computes $f : \{0,1\}^* \to \{0,1\}$ if C_n computes f^n for every n.

Circuit size and complexity

Definition: [size of a circuit]

The size of a circuit is the total number of gates (*counting the input variables as gates too*) in the circuit.

Definition: [size of a circuit family]

The size of a circuit family $C = (C_0, C_1, C_2, \ldots)$ is a function $s(\cdot)$ such that $s(n)$ is the size of C_n .

Definition: [circuit complexity]

The circuit complexity of a decision problem is the size of the minimal circuit family that decides it.

(This is the intrinsic complexity with respect to circuit size)

Poll

Let $f : \{0,1\}^* \to \{0,1\}$ be the parity decision problem.

$$f(x) = x_1 + \ldots + x_n \mod 2 \qquad \text{(where } n = |x|\text{)}$$

$$f(x) = x_1 \oplus \cdots \oplus x_n$$

What is the circuit complexity of this function?

Choose the tightest one:

$O(n)$

$O(n^2)$

$O(n^{2.5})$

None of the above.

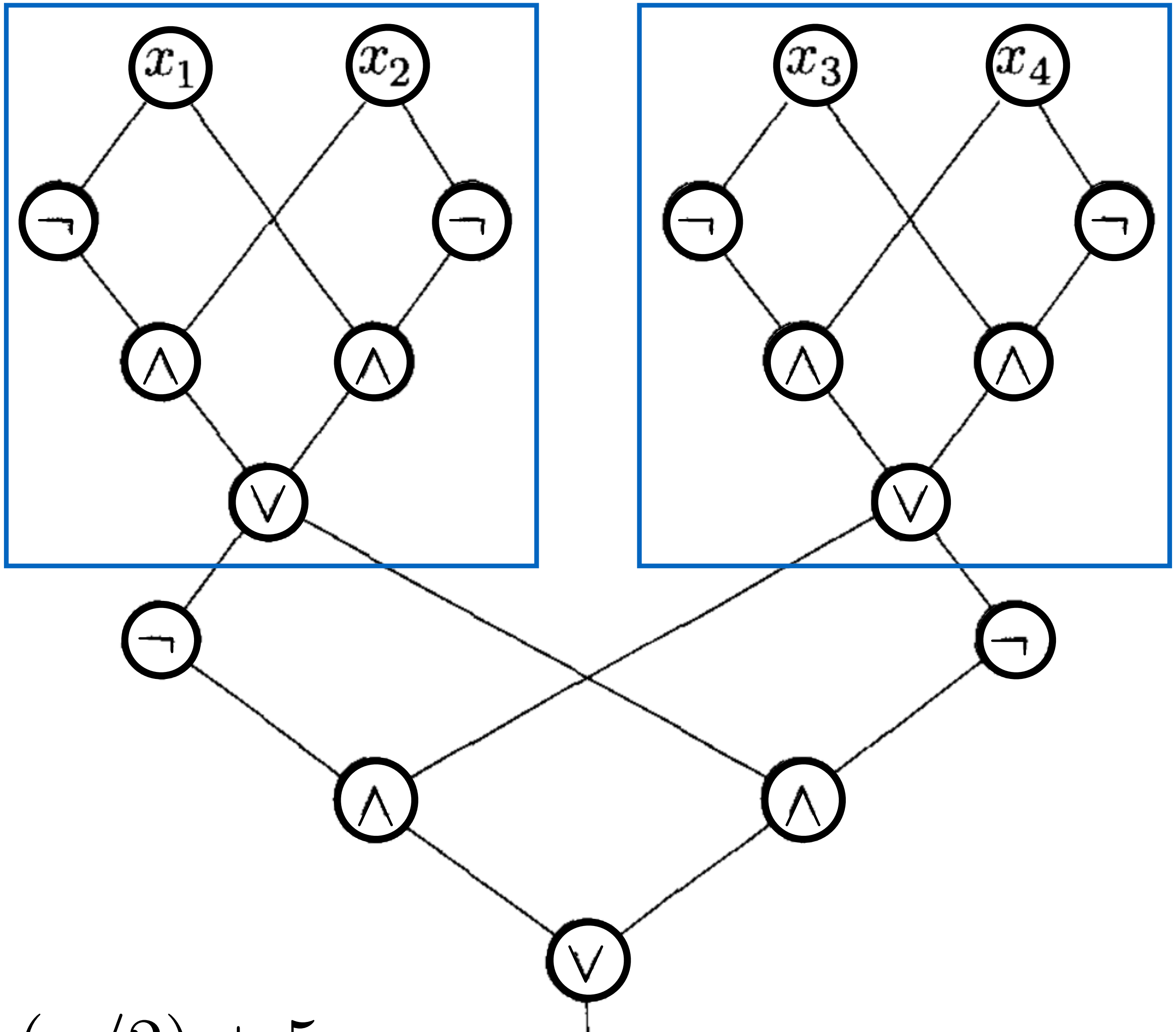
$O(2^n)$

$O(2^{2^n})$

$O(2\text{STACK}(n))$

Beats me.

Poll



$$s(n) = 2s(n/2) + 5$$
$$s(1) = 1$$

$$\Longrightarrow \; s(n) = O(n).$$

The big picture

Computability with respect to circuits

Theorem: Any decision problem $f : \{0,1\}^* \to \{0,1\}$ can be computed by a circuit family of size $O(2^n)$.

The big picture

Limits of efficient computability with respect to circuits

Theorem: There exists a decision problem such that any circuit family computing it must have size at least $2^n/4n$.

In fact, **most** decision problems require exponential size.

The big picture

Circuits can efficiently "simulate" TMs

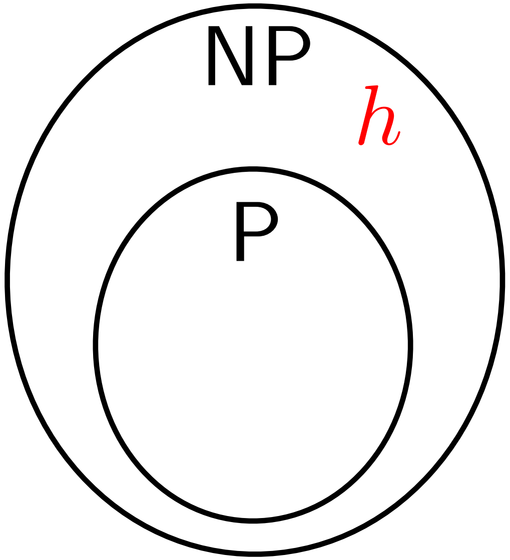
Theorem: Let $f : \{0,1\}^* \to \{0,1\}$ be a decision problem which can be decided in time $O(T(n))$.
Then it can be computed by a circuit family of size $O(T(n)^2)$.

poly-time TM $\implies$ poly-size circuits

no poly-size circuits $\implies$ no poly-time TM

The big picture

Circuits can efficiently "simulate" TMs



To show $\mathsf{P} \neq \mathsf{NP}$:

Find h in NP whose circuit complexity is more than poly(n).

The big picture

So we can just work with circuits instead

<u>This is awesome in 2 ways:</u>

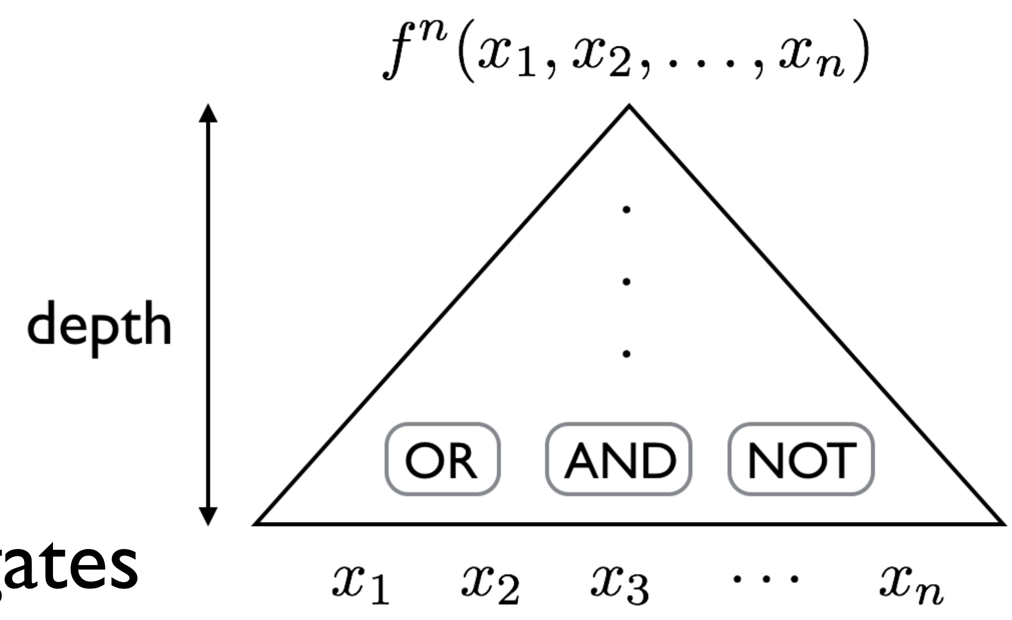
1. Circuits: clean and simple definition of computation.
"Just" a composition of AND, OR, NOT gates.

2. Restrict the circuit.

Make it less powerful.

e.g. (i) restrict depth

(ii) restrict types of gates



The big picture

So we can just work with circuits instead

Exciting progress was made in the 1980s.

People thought $\mathrm{P} \neq \mathrm{NP}$ would be proved soon.

Alas…

After 60 years of research,
best lower bound on circuit size for an explicit function:

$$5n - \text{peanuts}$$

The big picture

Theorem: Any decision problem $f:\{0,1\}^* \to \{0,1\}$ can be computed by a circuit family of size $O(2^n)$.

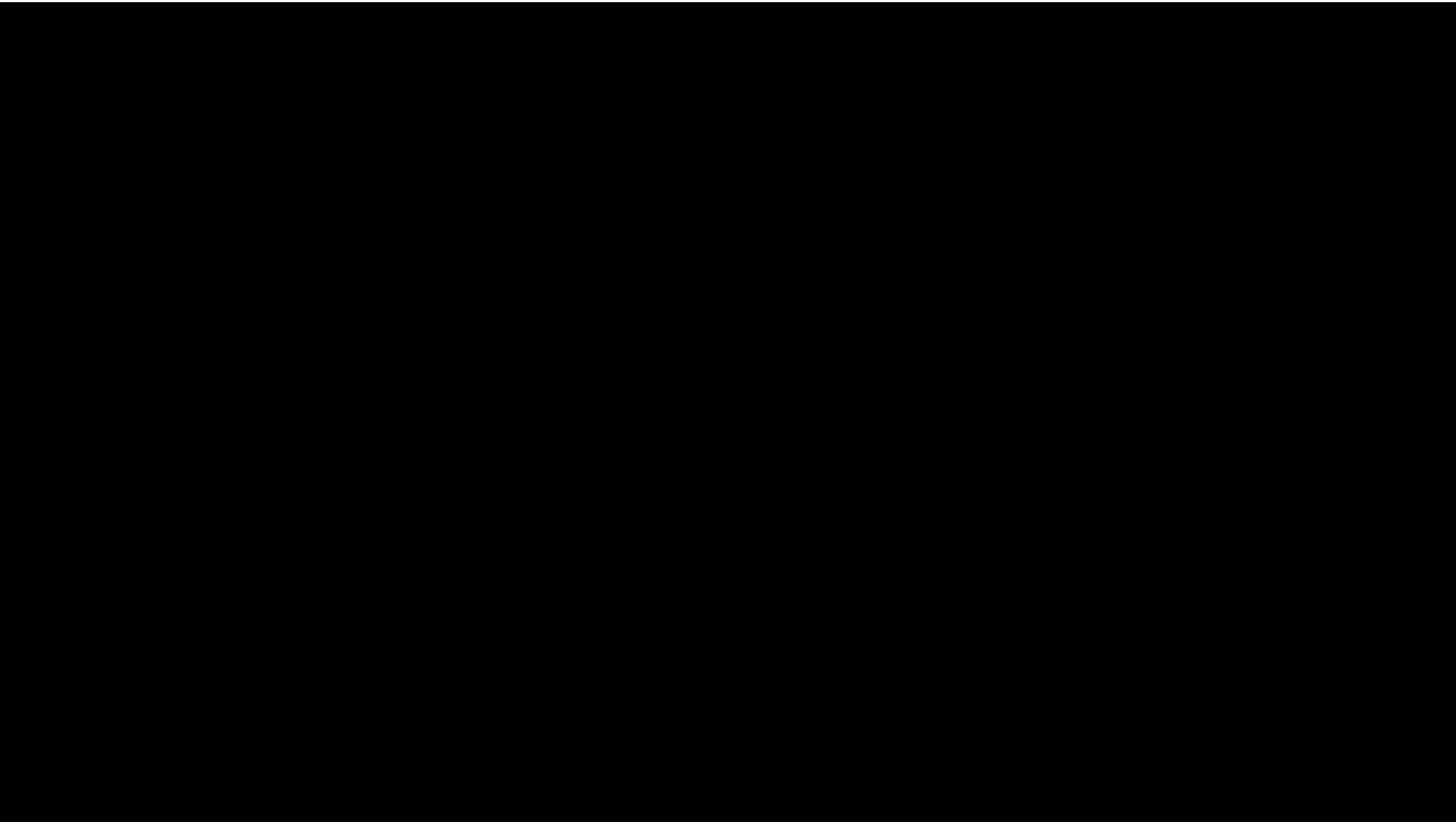
Theorem: There exists a decision problem such that any circuit family computing it must have size at least $2^n/4n$.

Theorem: Let $f:\{0,1\}^* \to \{0,1\}$ be a decision problem which can be decided in time $O(T(n))$.
Then it can be computed by a circuit family of size $O(T(n)^2)$.

A small break

Alan Turing
(1912 - 1954)

A small break



Theorem 1: Max circuit size of a function

Theorem: Any decision problem $f : \{0,1\}^* \to \{0,1\}$ can be computed by a circuit family of size $O(2^n)$.

Proof:

Goal:
construct a circuit of size $O(2^n)$ for $f^n : \{0,1\}^n \to \{0,1\}$.

Observation:

$$f^n(x_1, x_2, \ldots, x_n) = (x_1 \wedge f^n(1, x_2, \ldots, x_n)) \vee (\neg x_1 \wedge f^n(0, x_2, \ldots, x_n))$$

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if $x_1 = 1$: $x_1 \to 1$, $\neg x_1 \to 0$, second term $\to 0$

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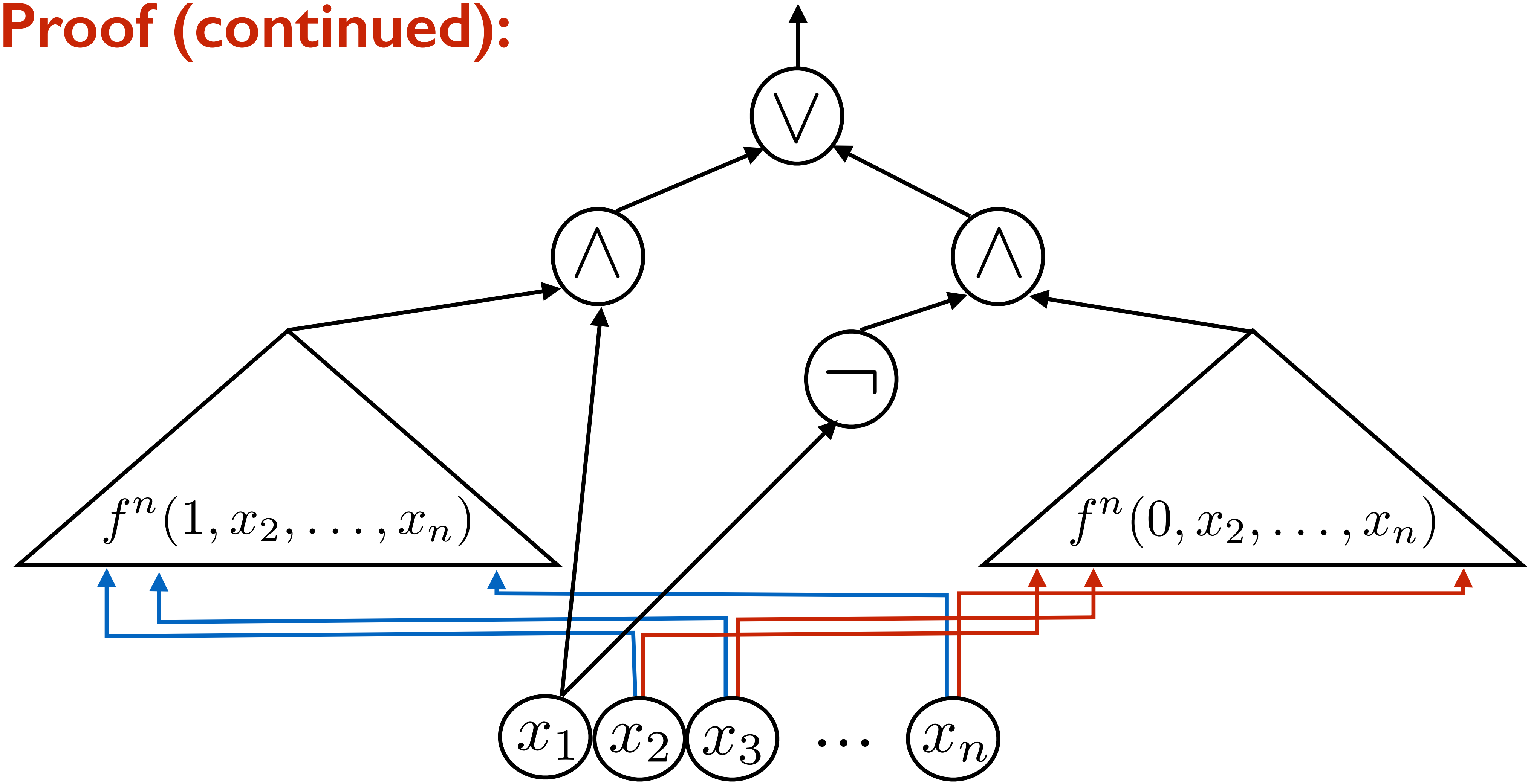
0 0

1

Theorem 1: Max circuit size of a function

Proof (continued):



$s(n) =$ max size of a circuit computing n-variable function

$$s(n) \leq 2s(n-1) + 5, \quad s(1) \leq 3 \implies s(n) = O(2^n) \quad \square$$

Poll

How many different functions $f : \{0,1\}^n \to \{0,1\}$ are there?

- n
- $2n$
- n^2
- 2^n
- 2^{2^n}
- $2\mathrm{STACK}(n)$
- none of the above
- beats me

Theorem 2: Some functions are hard

Theorem: There exists a decision problem such that any circuit family computing it must have size at least $2^n/4n$.

Proof:

Want to show: there is a function $f:\{0,1\}^n \to \{0,1\}$ that cannot be computed by a circuit of size $< 2^n/4n$.

Observation: # possible functions is 2^{2^n}.

Claim1: # circuits of size at most s is $\leq 2^{4s\log s}$.

Claim2: For $s \leq 2^n/4n$, $2^{4s\log s} < 2^{2^n}$.

$\#$ circuits $< \#$ functions

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We are done once we prove Claim 1. (Claim 2 is super easy.)

Theorem 2: Some functions are hard

Proof (continued):

Claim1: # circuits of size at most s is $\leq 2^{4s \log s}$.

Proof of claim:

Recall $|A| \leq |B|$ iff $B \twoheadrightarrow A$.

Let $A = \{\text{circuits of size at most } s\}$

$B = \{0,1\}^{4s \log s}$ $|B| = 2^{4s \log s}$

To show $B \twoheadrightarrow A$:

encode a circuit with a binary string of length $4s \log s$.

(just like the CS method)

Theorem 2: Some functions are hard

Proof (continued):

Claim1: # circuits of size at most s is $\leq 2^{4s \log s}$.

Proof of claim (continued):

Encoding a circuit with a binary string of length $4s \log s$:

Number the gates: 1, 2, 3, 4, …, s

For each gate in the circuit, write down:

- type of the gate (2 bits)
- from which gates the inputs are coming from (2 log s bits)

Total: s(2 + 2 log s) bits

(2s + 2s log s) bits ≤ (4s log s) bits □

Theorem 2: Some functions are hard

That was due to Claude Shannon (1949).

Father of *Information Theory*.

Claude Shannon
(1916 - 2001)

A non-constructive argument.

In fact, it is easy to show that **most** functions require exponential size circuits.

Theorem 3: Circuits can simulate TMs

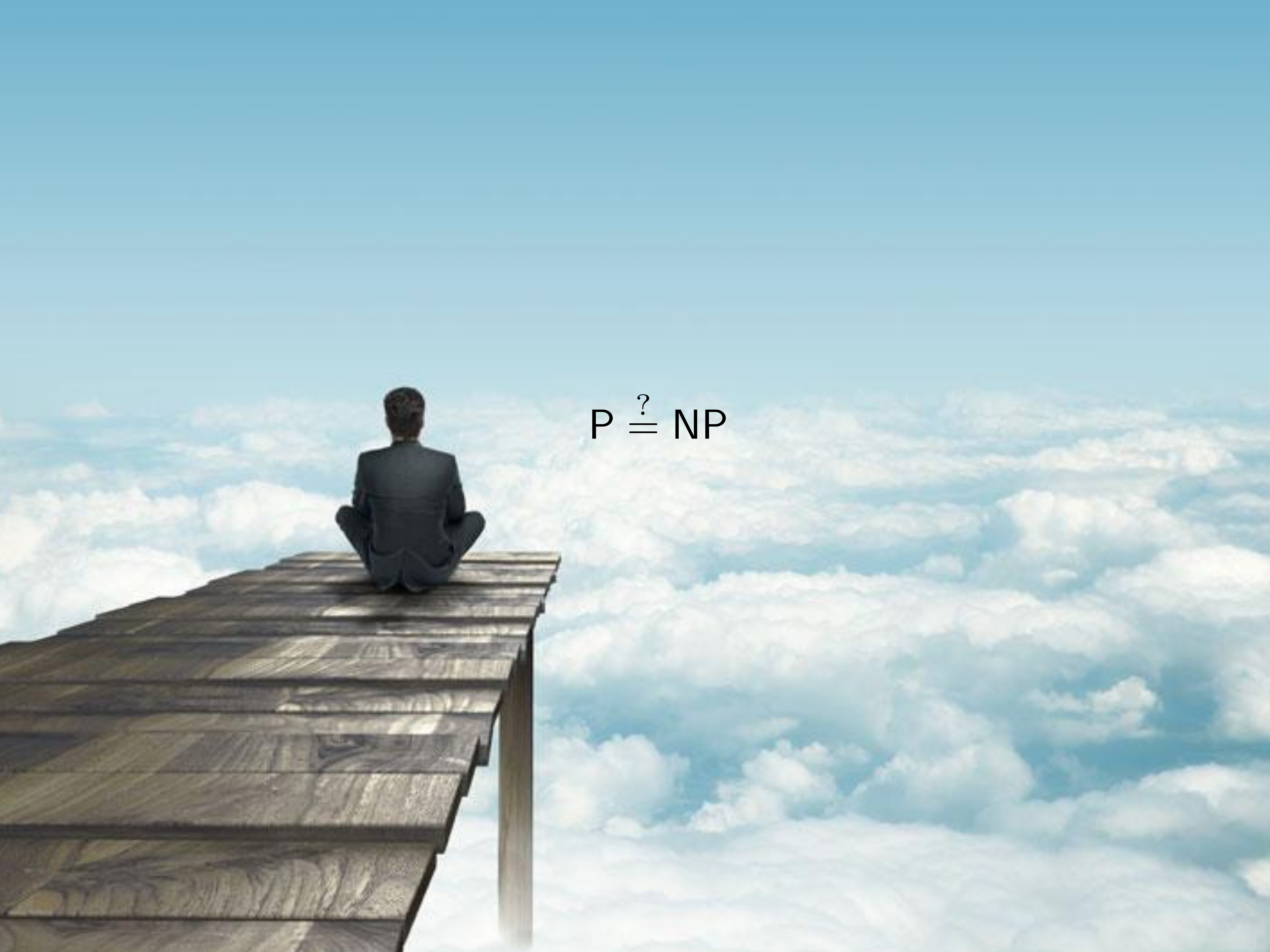
Theorem: Let $f : \{0,1\}^* \to \{0,1\}$ be a decision problem which can be decided in time $O(T(n))$.
Then it can be computed by a circuit family of size $O(T(n)^2)$.

How can you prove such a theorem?

If you like a challenge, try to prove it yourself.

If you don't like a challenge, but still curious,
see the course notes for a sketch of the proof.

If you don't like a challenge, and are not curious,
😞 you can ignore the proof.



$$\mathrm{P} \stackrel{?}{=} \mathrm{NP}$$