15-251: Great Theoretical Ideas In Computer Science

Recitation 12 Solutions

Rookie Mistake

A rook is placed in the lower left corner of an 8×8 chessboard. On each turn, the rook moves to a random legal location. The rook must move within its row or within its column and it cannot stay still. Let T denote the number of turns it takes until the rook lands in the upper right corner of the chessboard.

(a) Define the states and write the transition matrix for this Markov process.

(b) Find $\mathbf{E}[T]$.

The Markov chain only has 3 states. The first is the lower-left 7 by 7 block (including the start). The second is the rest of the board (top row and right column) **excluding** the top right square, and the third state is the top right corner (the goal).

(a) Call these states s_1, s_2, s_3. Let S_1, S_2, S_3 denote the set of positions on the chess board that correspond to s_1, s_2, s_3 respectively.

The probability of transitioning from s_1 to s_2 is $\frac{2}{14} = \frac{1}{7}$ because for any position in S_1, there is a $\frac{1}{14}$ chance of traversing to the topmost row and $\frac{1}{14}$ chance of traversing to the rightmost column, which both belong to s_2. Since we cannot get to s_3 from s_1, the probability of staying is $1 - \frac{1}{7} = \frac{6}{7}$.

The probability of traversing from s_2 to s_3 is $\frac{1}{14}$ because there is exactly one square that is the goal. There is a $\frac{6}{14}$ chance of staying in s_2 and $1 - \frac{1}{14} - \frac{6}{14} = \frac{1}{2}$ chance of going back to s_1. Starting from a position in S_3, we must go to a position in S_2. The transition matrix is

$$\begin{bmatrix} 6/7 & 1/7 & 0 \\ 1/2 & 3/7 & 1/14 \\ 0 & 1 & 0 \end{bmatrix}$$

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(b) Let $T_{1,3}$ and $T_{2,3}$ denote the number of moves it takes to get from s_1 to s_3 and from s_2 to s_3 respectively. using our Markov chain, we get the following relationships:

$$E[T_{1,3}] = \frac{1}{7}(E[T_{2,3}] + 1) + \frac{6}{7}(E[T_{1,3}] + 1)$$
$$E[T_{2,3}] = \frac{1}{14}(1) + \frac{3}{7}(E[T_{2,3}] + 1) + \frac{1}{2}(E(T_{1,3}] + 1)$$

Solving gives us $E[T_{1,3}] = 70$.

No directions

Let G be an undirected graph and consider the following random process. Start at the vertex 1, and at each step, pick and then move to a random neighbor of the current vertex.

(a) What does the transition matrix look like for this Markov chain?

(b) What is the stationary distribution?

Let $G = (V, E)$ be any undirected graph.

(a) At each vertex, we traverse to each of the neighbors uniformly at random. We can take each row of the adjacency matrix and divide the row by the degree of the vertex represented by that row (because there are that many neighbors). If K is the transition matrix, then $K_{u,v} = \frac{1}{\deg u}$ if $(u, v) \in E$ and $K_{u,v} = 0$ otherwise.

(b) A stationary distribution is a probability vector π satisfying $\pi K = \pi$ where K is the transition matrix. For a vertex v, let $\pi(v)$ denote the stationary probability associated to v. Then it must satisfy

$$\pi(v) = \sum_{u \in V} Pr[\text{go to v}|\text{we are at vertex u}]\pi(u) = \sum_{u \in N(v)} Pr[u \to v]\pi(u),$$

where $N(v)$ denotes the neighborhood of v. We claim that $\pi(v) = \frac{\deg(v)}{2|E|}$ satisfies this constraint, and we can check by substituting it into the above relation:

$$\frac{\deg(v)}{2|E|} = \sum_{u \in N(v)} \frac{1}{\deg(u)} \frac{\deg(u)}{2|E|} = \sum_{u \in N(v)} \frac{1}{2|E|} = \frac{\deg(v)}{2|E|}$$

because $|N(v)| = \deg(v)$.

Furthermore, notice that $\sum_{v \in V} \pi(v) = \frac{\sum_{v \in V}]deg(v)}{2|E|} = 1$, by the handshake lemma. So π is a valid probability vector. By uniqueness, this must be the stationary distribution.

General Communication

Let $F : X \times Y \to Z$ be a communication function. (In class, we focused on the case $X = Y = \{0, 1\}^n$ and $Z = \{0, 1\}$). Show that $\mathbf{D}(F) \leq \lceil \log_2 |X| \rceil + \lceil \log_2 |Z| \rceil$

We can encode any element in X with $\lceil \log_2(|X|) \rceil$ bits by enumerating the elements of X. We can do the same for any element in Y and Z. Therefore, if Alice and Bob want to compute $F(x, y)$ with $x \in X, y \in Y, F(x, y) \in Z$, Alice can send the encoding of x to Bob, and Bob can compute $F(x, y) = z$ and send the encoding of z to Alice. This takes $\lceil \log_2(|X|) \rceil + \lceil \log_2(|Z|) \rceil$ number of bits. Since we can do this with any function, this is an upper bound for $\mathbf{D}(F)$.

Rectangle

Let $F : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$ be a communication function and let M_F be the associated function matrix. Suppose that every monochromatic rectangle in M_F has size (number of entries) at most 2^n. What can you conclude about $\mathbf{D}(F)$?

Note that M_F is a $2^n \times 2^n$ sized function matrix so there are 2^{2n} entries. If every rectangle has size at most 2^n, then we need at least $\frac{2^{2n}}{2^n} = 2^n$ many rectangles to partition M_F.
Recall that any communication protocol of cost c partitions M_F into at most 2^c monochromatic rectangles. Since we need at least 2^n rectangles, this implies that $2^c \geq 2^n$ so $c \geq n$. This tells us that the protocol needs to cost at least n, so $\mathbf{D}(F) \geq n$.

Sharing randomization

Recall the randomized communication complexity model introduced in class. There, the players were allowed to individually flip coins, and make decisions based on the outcomes of those coin flips. We showed that $\mathbf{R}^\epsilon(\text{EQ}) = O(\log n)$, where the error probability $\epsilon = 1/n$. In this question we'll consider a slightly different randomized communication complexity model. In this new model, we'll assume that the players share a public coin. Whenever one of the players flips this coin, the other player automatically sees the outcome of the coin flip (without them communicating any bits). Show that in this model, there is a randomized protocol for EQ of cost $O(1)$ and error probability $1/2^{300}$. What is the exact relationship between the cost and the error probability of your protocol?

Suppose Alice and Bob wanted to know if their $n - bit$ strings x and y are equal. We generate a uniformly random n-bit string $r \in \{0,1\}^n$ by making n public coin flips and we will use the dot product operation. For two length n vectors, $\vec{a} \cdot \vec{b} = \sum_{i=1}^n a_i b_i$. Now, Alice will send the single bit $x \cdot r \mod 2$ to Bob, and Bob will let Alice know whether $x \cdot r \equiv y \cdot r \mod 2$. Notice that this protocol only uses 2 bits.

Next, we claim that if $x \neq y$ then the chance of $x \cdot r \equiv y \cdot r \mod 2$ is at most $\frac{1}{2}$. If $x \neq y$, then there exists an index i such that $x_i \neq y_i$. Without loss of generality, assume $i = n$. It's not completely obvious why we can make this assumption, but convince yourself that reindexing is okay. The proof still works if we don't make this assumption but the indices are messier.

Since $x \cdot r \equiv y \cdot r \mod 2$, we know that $\sum_{i=1}^n x_i r_i \equiv \sum_{i=1}^n y_i r_i \mod 2$. We will break the sum into two parts,

$$x_n r_n + \sum_{i=1}^{n-1} x_i r_i \equiv y_n r_n + \sum_{i=1}^{n-1} y_i r_i \mod 2$$

We can solve for r_n and get

$$r_n \equiv \frac{\sum_{i=1}^{n-1}(y_i r_i - x_i r_i)}{x_n - y_n} \mod 2.$$

Since $x_n \neq y_n$, the right hand side is well defined. We can also view the right hand side as fixed because r_n is chosen independently of all the other bits of r. Thus, $r_n = 0$ or $r_n = 1$ (but not both). The probability of picking the correct value for r_n is at most $\frac{1}{2}$ because we randomly pick 0 or 1. Thus, if $x \neq y$, then the chance of success is at most $\frac{1}{2}$.

To get the error bound of $1/2^{300}$, we would repeat this process 300 times and send 600 bits across the channels. In general, we would pay $2c$ for an error bound of $\frac{1}{2^c}$. The details are omitted.