15-251: Great Theoretical Ideas In Computer Science

Recitation 7

Proving that languages are in NP is not hard...

(a) A composite number is a number that is not a prime number and is not 1. Let COMPOSITE be the following decision problem: Given as input a positive integer N written in binary, is N composite? Prove that COMPOSITE $\in$ NP.

(b) Show that any language in NP can be decided in exponential time.

Properties of bipartite graphs

Show that G is bipartite if and only if it contains no cycles of odd length.

Assorted reductions between graph problems

HAMILTONIAN-CYCLE is the following problem: Given an undirected graph, is there a cycle that visits every vertex exactly once? HAMILTONIAN-CYCLE is NP-complete. Show that the following problems are NP-complete:

(a) Given a graph G, is there a path that visits every vertex of the graph exactly once?

(b) Given a graph G and an integer k, is there a path of length k?

(c) Given a graph G and an integer k, is there a spanning tree in G that contains at most k leaves?

...but proving that they're NP-complete is not easy.

The president of a large country can afford to build hospitals in up to k different towns. The goal is that everybody has a hospital in their town, or at least in a neighboring town. Show that determining if this is possible is NP-complete. More formally, let

$$\text{HOSP} = \{\langle G = (V, E), k\rangle : \exists H \subseteq V \text{ with } |H| \leq k \text{ such that } \forall v \in V, \\ \text{either } v \in H \text{ or } w \in H \text{ for some } \{v, w\} \in E\}.$$

Show that HOSP is NP-complete.

The NP-complete problem called VERTEX-COVER might be useful for this (we suggest that you reduce VERTEX-COVER to HOSP). VERTEX-COVER is defined as follows :

$$\text{VERTEX-COVER} = \{\langle G = (V, E), k\rangle : \exists C \subseteq V \text{ such that} \\ |C| = k \text{ and } \forall u, v \in E,\ (u \in C) \vee (v \in C)\}$$

Informally, the problem can be described as follows : given a graph G and a natural number k, is it possible to color exactly k vertices red such that every edge has at least one vertex colored red?