15-251: Great Theoretical Ideas In Computer Science

Recitation 5 Solutions

Gates has 3 floors

Show that any boolean function $f : \{0,1\}^n \to \{0,1\}$ (i.e. a function that takes in n input bits and outputs 1 bit) can be computed by a circuit of depth at most 3 (This means that the longest path from any input bit to the output should have at most 3 gates). Your gates may have any number of inputs. What is the size (in big-O) of such a circuit in the worst case?

We can express any boolean function as an AND of ORs (with possible negations). Thus, any input bit has to pass through at most one NOT, one AND and one OR gate, which means that H can be computed with a circuit of depth at most 3

Bounds on circuit size

Let $x_1, x_2, \ldots, x_n$ be input bits ($n \geq 2$). We use the convention that truth assignments are either 0 or 1. We are interested in computing the following Boolean function:

$$H(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if at least } 2 \text{ of the } x_i\text{'s are assigned } 1, \\ 0 & \text{if fewer than } 2 \text{ of the } x_i\text{'s are assigned } 1. \end{cases}$$

Prove there is a circuit computing H that uses at most $C \cdot n$ gates. Here C should be some fixed positive number, like 3 or 4 or 10. (Your C should work for every choice of n.) Your circuit can use any type of gate with fan-in at most 2 (though perhaps you will only need AND and OR gates?). If it helps you, you may assume that n is a power of 2.

[(**Bonus.** Lower bounds are hard. Prove that any circuit computing H must have at least $2n-3$ gates]

We claim that we can compute H for n input bits using at most $3n-4$ gates Define $O(x_1, x_2, \cdots, x_n)$ to be the function that's 1 iff at least 1 of the inputs is 1, and 0 otherwise.

We proceed by induction on the number of input bits, and at each step of induction, we compute both O and H - to strengthen the induction.

Base case. For 2 input bits, we simply OR the 2 bits to get the $O(x_1, x_2)$ and we AND them to get $H(x_1, x_2)$. $2 = 3 \cdot 2 - 4$, so our claim holds.

Induction. Assume we have a circuit of size at most $3n-4$ that computes H and O for n input bits. Let $H(x_1, \cdots, x_n) = h$ and $O(x_1, \cdots, x_n) = o$.

Consider an additional input (call it x_{n+1}).

Note that $H(x_1, \cdots, x_{n+1}) = (o \wedge x_{n+1}) \vee h$, and $O(x_1, \cdots, x_{n+1}) = o \vee x_{n+1}$.

Thus, we can extend the circuit for $n+1$ input bits using only 3 additional gates, meaning the upper bound for $n+1$ bits is $3(n+1)-4$.

Degrees and Paths

Suppose that a graph G has minimal degree d (so the vertex with the smallest degree has degree d). Show that G has a path of length d.

Suppose that the claim is not true for some G. Consider a maximal path P and let v be an endpoint. Since P has at most d edges, it has at most d vertices, with one of them being v. Also note that v has at least d neighbors - this means that there exists at least one neighbor u of v that does not appear in P. But then attaching the edge $\{v, u\}$ to P gives a larger path, contradicting our assumption that P was maximal.

Useful tree facts

(a) Let T be a tree with at least two vertices. Prove that T has at least two leaves.

Suppose that T does not have at least two leaves. Then either there are exactly $n-1$ internal nodes and a single leaf, or there are exactly n internal nodes. Internal nodes have degree at least 2, so the sum of the degrees of vertices in T is at least

$$\min\{2(n-1)+1, 2n\} = 2n-1$$

Also, by T being a tree, $|E| = n-1$. Then the handshake lemma

$$2|E| = \sum_{v \in V} \deg(v)$$

gives $2(n-1) \geq 2n-1$ which is a contradiction.

(b) Let $G = (V, E)$ be a graph. Show that G is a tree if and only if for every pair of distinct vertices $u, v \in V$ there is a unique path in G from u to v.

Suppose that G is a tree. Let $u, v \in V$ be distinct vertices. Note that since G is a tree, it is connected, so there is at least one path between u and v. Now assume that there are two distinct paths between these vertices, say

$$u = a_0, a_1, ..., a_s = v$$

and

$$u = b_0, b_1, ..., b_t = v$$

As these are distinct paths, there exists smallest index i such that $a_i \neq b_i$ (note $i \geq 1$). Also, since $a_s = b_t$, there must exist smallest index j larger than i such that $w_j = w_k$ for some $k > i$ as well. Then

$$a_{i-1}, a_i, ..., a_j = b_k, b_{k-1}, ..., b_i, b_{i-1} = a_{i-1}$$

is a cycle. It is a walk whose endpoints coincide and has no repeating edges. This contradicts the assumption that G is a tree.

Suppose that there exists a unique path between any two distinct vertices of G. Immediately by definition, this implies that G is connected. Next, we show that G is acyclic. If G has a cycle, say

$$v_0, v_1, ..., v_k = v_0$$

where $k \geq 2$, then

$$v_0, v_1$$

and

$$v_k, v_{k-1}, ..., v_2, v_1$$

are two distinct paths from v_0 to v_1 which contradicts the assumption.