15-251: Great Theoretical Ideas In Computer Science

Recitation 4 Solutions

$\mathcal{O}$, I Think I Understand Asymptotics Now

Let f, g, h be functions from $\mathbb{N}$ to $\mathbb{N}$. Prove or disprove the following:

(a) If $f \in \mathcal{O}(g)$ and $g \in \mathcal{O}(h)$, then $f \in \mathcal{O}(h)$

We know by definition that there exist n_1 and c_1 such that for all $n \geq n_1$, $f(n) \leq c_1 g(n)$.
Similarly, we have n_2 and c_2 for g and h.
Choose $n_0 = \max(n_1, n_2)$ and $c = c_1 c_2$ and let $n \geq n_0$.
Then, $f(n) \leq c_1 g(n) \leq c_1 c_2 h(n) = ch(n)$, as desired, so $f \in \mathcal{O}(h)$

(b) If $f \in \mathcal{O}(g)$, then $g \in \mathcal{O}(f)$

Let $f(n) = 1$ and $g(n) = n$.
While $f \in \mathcal{O}(g)$, there is no n_0, c you can pick that makes $n \leq c$ for all $n \geq n_0$, as we just let $n > \max(c, n_0)$ as a counterexample.
This seems like a silly and obvious example, but remember to be rigorous when disproving big-O membership: generally you'll have to find a counterexample in terms of arbitrary proposed c, n_0.

(c) For all $k \in \mathbb{R}^+$, $\log(n) \in \mathcal{O}(n^k)$.

We choose $n_0 = 1$ and $c = \frac{1}{k}$, and let $n \geq n_0$ arbitrary.
Note that $\log(n^k) \leq n^k$, as this inequality is true for all positive values n^k.
By properties of logarithms, $k \log(n) = \log(n^k)$.
Rearranging our inequalities, we then have that $\log(n) \leq \frac{n^k}{k} = cn^k$, as desired, concluding the proof.

Odd-Paz

State and prove a divide-and-conquer procedure for proportional cake cutting between any number of players. (The Even-Paz algorithm as described in lecture is an excellent starting point.)

If $n = 1$: the single player takes the whole cake.
If $n = 2k$ for some $k \in \mathbb{N}^+$: every player draws a vertical line on the cake that cuts it in half according to their valuation function. Cut the cake anywhere between the kth and $(k+1)$th lines inclusive. The players who drew the k leftmost lines recurse on the left piece and the players who drew the k rightmost lines recurse on the right piece.
If $n = 2k+1$ for some $k \in \mathbb{N}^+$: every player draws a vertical line such that, according to their valuation function, the left side of the line is worth $\frac{k}{2k+1}$ of the total cake and the right side of the line is worth $\frac{k+1}{2k+1}$ of the total cake. Cut the cake on the median line. The players who drew the k leftmost lines recurse on the left piece and the players who drew the $k+1$ rightmost lines recurse on the right piece.

Claim: This algorithm is proportional: for any piece C of cake and for any number n of players, each player will walk away with a slice S_i that they feel is worth at least $\frac{1}{n}$ of the value of C.
(Equivalently, $\forall i.\ n \cdot V_i(S_i) \geq V_i(C)$). The proof is by induction.
Base case ($n = 1$): duh. The player walks away with all of the remaining cake. $1 \cdot V_i(C) \geq V_i(C)$.
Induction hypothesis: for some $k \in \mathbb{N}^+$, this algorithm always achieves a proportional allocation among $(k-1)$ players.
Induction step ($n = 2k$): Note that after the cake is cut, the k players who recurse on the left piece (call it L) value it at least half as much as they value C. By the IH, each player i will end up with a piece that they think is worth at least $\frac{V_i(L)}{k}$. But since we just showed that $V_i(L) \geq \frac{V_i(C)}{2}$ for these players, this value is at least $\frac{V_i(C)}{n}$. So proportionality is achieved for these k players. An identical argument applies to the other k players, who recurse on the right piece.
Induction step ($n = 2k+1$): Note that after the cake is cut, the k players who recurse on the left piece (call it L) value it at least $\frac{k}{2k+1}$ as much as they value C. By the IH, each player i will end up with a piece that they think is worth at least $\frac{V_i(L)}{k} \geq \frac{V_i(C)}{2k+1} = \frac{V_i(C)}{n}$. A structurally identical argument applies to the $k+1$ players who recurse on the right piece: by the IH, they each end up with a piece that they think is worth at least $\frac{V_i(R)}{k+1} \geq \frac{V_i(C)}{2k+1} = \frac{V_i(C)}{n}$. So proportionality is achieved for everybody.