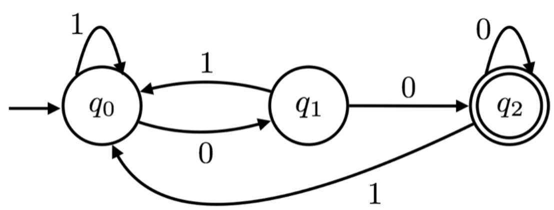
1 Short

1. How many elements does the set $\{a, b, c\}^3$ have?

2. Let $\Sigma = \{0, 1\}$ and let $L = \{w \in \{0, 1\}^* : |w| \text{ is even}\}$. Define the decision problem corresponding to L.

3. For the DFA below, write down its transition function.



4. Let L be a language. Write the definition of "L is decidable".

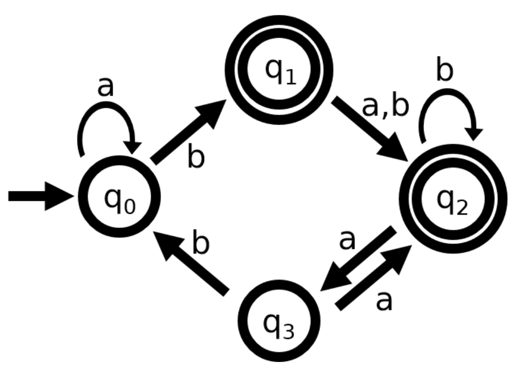
5. Find a counter-example to the following claim: Given any two functions $f(n)$ and $g(n)$, either $f(n) = O(g(n))$ or $g(n) = O(f(n))$.

6. Prove that if $f(n) = O(g(n))$ then $g(n) = \Omega(f(n))$.

7. Let U be the set of all languages over the alphabet $\Sigma = \{1\}$. Circle the correct statements, and cross out the incorrect ones:

   U is finite U is countable U is uncountable

8. Consider the following DFA called M:



   "aaba $\in L(M)$": True or False? ____________________

   "$L(M)$ contains a string of length 100": True or False? ____________________

9. What is the Church–Turing thesis?

10. For a language L, we define $L^* = \{x_1 x_2 \dots x_k : k \geq 0, \forall i\ x_i \in L\}$. Consider the following claim: If L_1 and L_2 are languages over the alphabet $\{a, b\}$ then $(L_1 \cap L_2)^* = L_1^* \cap L_2^*$. Is this claim True or False? If False, you need to provide a counterexample. If True, no justification is required.

2 Medium to Long Answer

1. Prove that n^3 is not $O(n^2)$.

2. Let $\Sigma = \{0, 1\}$, and let L be the set of all words that contain 100 or 110 as a substring. Draw the state diagram of a DFA that decides L.

3. Let $\Sigma = \{a, b\}$. Draw the state diagram of a DFA that decides the language

$$L = \{a^n b^m : n, m \in \mathbb{N}^+,\ n \equiv m \mod 3\}.$$

4. Let $\Sigma = \{a, b\}$. Draw the state diagram of a TM that decides the language

$$L = \{w : w \text{ contains the same number of } a\text{'s and } b\text{'s}\}.$$

   Clearly define the tape alphabet you are using.

5. Suppose that $L_1, L_2 \subseteq \{0, 1\}^*$ are decidable languages. Show that $L_1 \cdot L_2$ is also decidable. (Recall that $L_1 \cdot L_2$ is the language of all strings of the form xy, where $x \in L_1$ and $y \in L_2$.) You may use pseudocode when justifying your solution. If you feel your pseudocode is "obviously" correct, you may assert this (but make sure it's definitely correct!).

6. Assume the languages L_1 and L_2 are in P. Prove or give a counter-example: $L_1 \cap L_2$ is in P.

7. Let Let $\Sigma = \{a, b\}$. Show that the language $L = \{www : w \in \{a, b\}^*\}$ is not regular.

8. Show that the set of all finite subsets of $\{0, 1\}^*$ is countable.

9. Draw a circuit that computes the 3-variable function $\text{MAJ} : \{0, 1\}^3 \to \{0, 1\}$, which is defined as $\text{MAJ}(x_1, x_2, x_n) = 1$ iff $x_1 + x_2 + x_3 \geq 2$. For full credit, the size of your circuit must be at most 9.

10. For which $n \in \mathbb{N}^+$ is it possible to have an n-vertex graph G in which all vertices have distinct degrees?

11. Show that in the cake cutting problem, if an allocation is envy-free, then it is proportional.

12. Let $L \subseteq \{0, 1\}^*$ be a a finite language (i.e., L contains only finitely many strings). Prove that L is regular.

13. Prove Cantor's theorem: for any non-empty set A, $|\mathcal{P}(A)| > |A|$.

14. Recall that a set A is called *countably infinite* if it is infinite and $|A| \leq |\mathbb{N}|$. Show that A is countably infinite if and only if $|A| = |\mathbb{N}|$.

15. Suppose we define the language

$$G = \{\langle M \rangle : M \text{ is a TM such that the number of strings } M \text{ accepts is exactly one}\}.$$

Show that G is undecidable.

16. Consider the following piece of pseudocode M, which takes a natural number n as input:

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M(n):
    Let t = 0.
    While n ≠ 1:
        Let t = t + 1.
        If n is even then let n = n/2, else let n = 3n + 1.
    End While.
    If t is even then ACCEPT, else REJECT.
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You have a burning desire to know whether M halts when run on $n = 63728127$. We strongly expect you will not be able to prove this one way or the other during the exam.

However, suppose that you are allowed to send one email to Prof. Ada, containing some pseudocode M_{AA} for a function that takes no inputs and either ACCEPTS or REJECTS (or loops forever, of course). And Prof. Ada will reply telling you correctly whether or not M_{AA} ACCEPTS. You can also send one email to Prof. Procaccia, containing some pseudocode M_{AP} for a function that takes no inputs and either ACCEPTS or REJECTS (or loops forever, of course). He will also reply telling you whether or not M_{AP} ACCEPTS.

Explain how you can easily determine whether $M(63728127)$ halts or not.

17. Show that if a graph $G = (V, E)$ is acyclic and satisfies $|E| = |V| - 1$, then it is connected.

18. Let $T(n)$ satisfy the following recurrence relation:

$$T(1) = 0, \qquad T(n) = 4 \cdot T(n/4) + 3 \cdot n^2 \quad \text{for } n > 1,$$

You can assume n is power of 4, i.e. $n = 4^t$ for some $t \in \mathbb{N}$.

   (a) Consider the recursion tree corresponding to the above recursive relation. Determine the total number of nodes in the tree, in terms of n, using the $\Theta(\cdot)$ notation.

   (b) Determine $T(n)$ using the $\Theta(\cdot)$ notation.

19. Let $\Sigma = \{0, 1\}$ and $L = \{w \in \{0,1\}^* : w \text{ contains no 1's}\}$. Show that the circuit complexity of L is $\Omega(n)$.

20. Design a cake cutting algorithm for a set of players $N = \{1, \ldots, n\}$ that finds an allocation A such that for all $i = 1, \ldots, n-1$ (all players except n), $0 < V_i(A_i) \leq \epsilon$ (*at most* ϵ), for a given $\epsilon > 0$. Analyze the complexity of your algorithm in the Robertson-Webb model. You may assume that for any two *distinct* points $x, y \in [0, 1]$, and any player $i \in N$, $V_i([x, y]) > 0$, that is, each player has a strictly positive value for any interval that is not a single point.