15-251: Great Theoretical Ideas In Computer Science

Fall 2013

Test Solutions

Part I: Short answers (40 points)

Even though the model solutions provide an explanation for each answer, you do not need to give such explanations in the exam.

1. [5 points]
Recall that a "Manhattan Walk" is a walk on a grid that at each step either goes right or up.
How many Manhattan Walks are there from (x_1, y_1) to (x_2, y_2), where $x_1 \leq x_2$ and $y_1 \leq y_2$?

Solution: $\binom{y_2 - y_1 + x_2 - x_1}{x_2 - x_1}$
Explanation: Out of the total $(y_2 - y_1 + x_2 - x_1)$ UP+RIGHT steps that need to be taken, determine which $x_2 - x_1$ will be RIGHT (and the rest will be UP).

2. [5 points]
Is the group $(Z_7, +)$, i.e. $\{0, 1, 2, \ldots, 6\}$ under addition modulo 7, cyclic? If so, how many generators does the group have?

Solution: Yes. 6 generators (all except 0).

Explanation: The generators of $(Z_n, +)$ are the integers coprime with n.

3. [5 points]
Suppose a planar graph $G = (V, E)$ has a planar drawing such that every face has at least 4 edges. Argue why G must have a vertex of degree at most 3.

Solution: Suppose for contradiction that every vertex has degree at least 4. Then, we have $E \geq 4V/2 = 2V$. We are also given that every face has at least 4 edges, and every edge is shared between two faces. Therefore, $2E \geq 4F$, i.e., $E \geq 2F$. Thus,

$$V - E + F \leq \frac{E}{2} - E + \frac{E}{2} = 0 - F < 2.$$

Hence, G does not satisfy Euler's formula, a contradiction. Thus, G must have a vertex of degree at most 3.

4. [5 points]
How many polynomials in $F_3[x]$ have degree at most 3?

Solution: $3^4 = 81$.

Explanation: Polynomials in $F_3[x]$ of degree at most 3 are of the form $c_3x^3 + c_2x^2 + c_1x + c_0$, where each $c_i \in \{0, 1, 2\}$.

5. [5 points]
Calculate

$$108^{(7^{108})} \text{ mod } 35.$$

Solution: 3

Explanation: Note that $\phi(35) = 24$, and $\phi(24) = 8$. Now,

$$\begin{aligned} 108^{(7^{108})} \text{ mod } 35 &= 108^{(7^{108} \text{ mod } 24)} \text{ mod } 35 = 108^{(7^{108 \text{ mod } 8} \text{ mod } 24)} \text{ mod } 35 \\ &= 108^{(7^{4} \text{ mod } 24)} \text{ mod } 35 = 108^{1} \text{ mod } 35 = 3. \end{aligned}$$

6. [5 points]
In the Robertson-Webb model of cake cutting algorithms, what is the answer to $\text{cut}_i(0, \text{eval}_i(0, \frac{1}{\pi}))$?

Solution: $\frac{1}{\pi}$.

Explanation: If $c = \text{eval}_i(0, \frac{1}{\pi})$, then $\text{cut}_i(0, c)$ asks for the endpoint x such that the value of $[0, x]$ is c for player i. Clearly, $x = 1/\pi$.

7. [5 points]
I throw a $(k-1)$-sided die (with sides labeled $1, \ldots, k-1$) n times, and get $1 every time the result is in Z_k^*. What is the expected number of dollars that I get, as a function of n and $\phi(k)$?

Solution: $n \cdot \phi(k)/(k-1)$.

Explanation: The chance of getting a number in Z_k^* is $\phi(k)/(k-1)$. Use linearity of expectation.

8. [5 points]
Prove or disprove: In an election with two alternatives, Borda count is Condorcet consistent.

Solution: True. In an election with two alternatives, the Borda score of an alternative is the number of times it is preferred over the other alternative. If there exists a Condorcet winner, it would have score more than $n/2$ (where n is the number of voters), and the other alternative would have score less than $n/2$. Hence, a Condorcet winner would always win.

Part II: Homework problem (10 points)

The actual exam will include a homework problem.

Part III: Longer Problems (50 points)

9. [10 points]
Prove that any finite group G which contains an even number of elements contains an element of order 2.
Hint: Let $t(G) = \{g \in G \mid g \neq g^{-1}\}$. Show that $t(G)$ has an even number of elements and every non-identity element of $G - t(G)$ has order 2.

Solution: Let $t(G) = \{g \in G \mid g \neq g^{-1}\}$. Note that $g \in t(G)$ if and only if $g^{-1} \in t(G)$. Since $g \neq g^{-1}$ for every $g \in t(G)$, we can pair up every element in $t(G)$ with its inverse. This shows that $|t(G)|$ must be even. Since $|G|$ is also even, we get that $|G \setminus t(G)|$ must also be even.

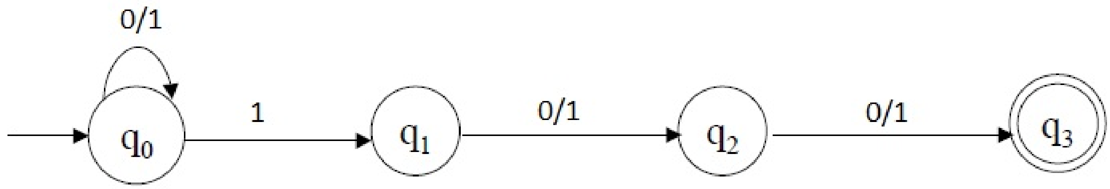
Note that for the identity element e of G, $e \notin t(G)$ since $e = e^{-1}$. Therefore, $|G \setminus t(G)| > 0$, so $|G \setminus t(G)| \geq 2$. Thus, there exists an element $l \in G \setminus t(G)$ such that $l \neq e$. By definition of $t(G)$, $l = l^{-1}$, so the order of l is at most 2. However, $l \neq e$, and e is the unique element with order 1. Hence, order of l must be 2.

10. [10 points]
For an integer $n \geq 1$, define the language

$$L_n = \{w \in \{0,1\}^\star \mid \text{ the } n\text{'th-to last symbol in } w \text{ is a } 1\} \ .$$

Draw an NFA that accepts L_3.

Solution:



11. [10 points]
Consider the following problem: `15251-COLOR`

INSTANCE: Simple undirected graph $G = (V, E)$.

QUESTION: Does there exist a proper coloring of G using at most 15251 colors? (A proper coloring colors the vertices so that no edge has two endpoints of the same color.)

Prove that `15251-COLOR` is NP-complete.

Solution: First, we show that `15251-COLOR` is in NP. Given a 15251-coloring of the input graph, one could iterative over the edges, check if the two endpoints have different colors, and thus verify the solution in polynomial time. Hence, `15251-COLOR` is in NP.

We show that `15251-COLOR` is NP-hard by showing a reduction from `3-COLOR`, which is known to be NP-complete. Given an instance $G = (V, E)$ of `3-COLOR`, for the graph G' by adding a 15248-clique to G, and connecting each vertex of the added clique to each vertex of G. We show that G is 3-colorable if and only if G' is 15251-colorable. Given any 3-coloring of G, one could easily obtain a 15251-coloring of G' by simply coloring each of the 15248 added vertices with a new color. On the other hand, suppose there exists a 15251-coloring of G'. Since the added 15248 vertices form a clique, and each of them is connected to every vertex in G, they must use 15248 distinct colors, which are further distinct from the colors of vertices of G. Hence, vertices of G must be colored with at most 3 colors, giving a 3-coloring of G. Thus, we have shown that `3-COLOR` $<_P$ `15251-COLOR`. This shows that `15251-COLOR` is NP-hard.
Thus, `15251-COLOR` is NP-complete.

12. [10 points]

If $p \geq 2$ is a prime, a p-onic number is any number x of the form

$$x = \frac{a_1}{p} + \frac{a_2}{p^2} + ... = \sum_{k=1}^{\infty} \frac{a_k}{p^k}$$

where each $a_k \in Z_p$. Prove that the set of p-onic numbers is uncountable.

Solution: For any number $x \in [0, 1]$, looking at the p-ary representation of x shows that x must be a p-onic number. Hence, the set of p-onic numbers contain the uncountable set of numbers $[0, 1]$, and thus must be uncountable itself.

Alternatively, one could also use Cantor's argument.

13. [10 points]

Suppose you randomly color the vertices of the complete graph on n vertices one of k colors. What is the expected number of paths of length $c \geq 3$ such that no two adjacent vertices on the path have the same color?

Solution: Note that the number of paths of length c is $t = \frac{1}{2} \cdot n \cdot (n-1) \cdot \ldots \cdot (n-c+1)$. The factor $1/2$ is present because every path is counted twice, once from each endpoint. For $i \in [1, t]$, let P_i be random variables denoting if no two adjacent vertices in the i^{th} path have the same color. Then,

$$\mathbb{E}[P_i] = 1 \cdot \frac{k-1}{k} \cdot \ldots \frac{k-1}{k} = \left(\frac{k-1}{k}\right)^{c-1}.$$

This can be observed by sequentially coloring the vertices of the path starting from any one end. Except for the first vertex, every vertex can be colored with all but one color (the color of its previous vertex). Using linearity of expectation,

$$\mathbb{E}[P] = \sum_{i=1}^{t} \cdot \mathbb{E}[P_i] = \frac{1}{2} \cdot \left(\prod_{i=0}^{c-1} (n-i)\right) \cdot \left(\frac{k-1}{k}\right)^{c-1}.$$