15-251: Great Theoretical Ideas In Computer Science

Fall 2013

Practice Midterm Test 1

Name:

Andrew ID:

Section:

INSTRUCTIONS:

- **Write your NAME, ANDREW ID, and SECTION above**.
- You may refer to your 3-by-5 index card, but no other notes. You may *not* use a calculator.
- Except for Part I (short answers), you must give clear and complete proofs. In your proofs, you may quote and use any results presented during lectures (unless specified otherwise); all other steps and results REQUIRE PROOFS. **Please write clearly**.

Problem	Points	Score	Problem	Points	Score
1	10		4	20	
2	10		5	25	
3	10		6	25	
			Total	100	

Part I: Short answers ($3 \times 10 = 30$ points)

This part is to test your understanding of the basic concepts emphasized in class or recitations. You should be able to solve these problems using facts and concepts you know from memory, possibly applied with simple twists.

This part is meant to be relatively quick, but be sure to take enough time to read the questions carefully and calmly deduce your (hopefully correct) answers!

You do **not** need to give justification for your answers in this part (except when the question asks for it). However, if you do not know the answer, show your work to (possibly) get partial credit.

1. [10 points]
There are 8 envelopes. One of them has a $100 bill in it. You start by picking a random envelope (each with probability $1/8$). If it has the $100 bill, you take it and go home. If it does not, you throw it away, and pick one of the remaining 7 envelopes at random. You countinue this process until you find the $100 bill. What is the probability that you will find the bill in exactly the 5^{th} iteration?

2. [10 points]
What lower bound does the "breaking-apart" argument give on P_n for a stack of n pancakes ($n > 3$)? This is in the standard model where all pancakes have different sizes and you can flip any number of pancakes from the top in one operation.

3. [10 points]
How many non-negative integer solutions $(x_1, x_2, \ldots, x_5)$ are there to

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 251 \ ?$$

Your answer must be in the form of a *single binomial coefficient*.

Part II: (Variant of) Homework Question (20 points)

Note that even though the following question does not introduce any variation from its homework version, the question in the exam might.

4. [20 points]

n cars are standing at arbitrary points on a circle. Each car has a certain amount of fuel in it. Once a car a reaches another car b, a can take all the fuel that b has, and move forward. There is exactly enough fuel among all n cars to fuel any single one of them to make a full circle. Prove that there is a car that can move around the circle clockwise, taking fuel from other cars as it passes them, and run out of fuel just as it reaches its starting point.

Part III: Longer Problems ($2 \times 25 = 50$ points)

Remember that in this part, you need to formally prove your answer for full credit. Be crisp, precise, and accurate!

5. [25 points]
A company Toca Cola(!) is holding a contest. They have issued n types of coupons, and everyone who collects at least one coupon of each type wins a prize. You get a coupon of a random type with each purchase of a Toca Cola. What is the expected number of Toca Colas you will need to buy in order to collect all the coupons? Your answer can be in the form of a summation, but its terms can only involve constants, variable(s) of the summation, and n.

6. [25 points]

How many spanning trees (not necessarily minimum) does $K_{2,n}$ have? Recall that $K_{a,b}$ is the complete bipartite graph with a vertices on one side and b vertices on the other. You must give a closed form formula as a function of n for full credit.