15-251: Great Theoretical Ideas In Computer Science

Fall 2013

Practice Test Solutions 1

Problem	Points	Score	Problem	Points	Score
1	10		4	20	
2	10		5	25	
3	10		6	25	
			Total	100	

Part I: Short answers ($3 \times 10 = 30$ points)

This part is to test your understanding of the basic concepts emphasized in class or recitations. You should be able to solve these problems using facts and concepts you know from memory, possibly applied with simple twists.

This part is meant to be relatively quick, but be sure to take enough time to read the questions carefully and calmly deduce your (hopefully correct) answers!

You do **not** need to give justification for your answers in this part (except when the question asks for it). However, if you do not know the answer, show your work to (possibly) get partial credit.

THE JUSTIFICATION PROVIDED IN THIS PART IS ONLY FOR YOUR UNDERSTANDING. YOU WOULD NOT NEED TO PROVIDE A JUSTIFICATION IN THE EXAM UNLESS IT IS ASKED FOR IN THE QUESTION.

1. [10 points]
There are 8 envelopes. One of them has a $100 bill in it. You start by picking a random envelope (each with probability $1/8$). If it has the $100 bill, you take it and go home. If it does not, you throw it away, and pick one of the remaining 7 envelopes at random. You countinue this process until you find the $100 bill. What is the probability that you will find the bill in exactly the 5^{th} iteration?

ANSWER: $1/8$.

JUSTIFICATION: The probability that you choose a wrong envelope in the first round $(7/8)$, a wrong envelope among the remaining ones in the second round $(6/7)$, and so on until you select the right envelope among the remaining ones in the fifth round $(1/4)$ is $7/8 \times 6/7 \times 5/6 \times 4/5 \times 1/4 = 1/8$.

2. [10 points]
What lower bound does the "breaking-apart" argument give on P_n for a stack of n pancakes $(n > 3)$? This is in the standard model where all pancakes have different sizes and you can flip any number of pancakes from the top in one operation.

ANSWER: $P_n \geq n$.

3. [10 points]
How many non-negative integer solutions $(x_1, x_2, \ldots, x_5)$ are there to

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 251 \text{ ?}$$

Your answer must be in the form of a *single binomial coefficient*.

ANSWER: $\binom{256}{5}$.

JUSTIFICATION: Imagine 251 objects in a row. You want to put five dividers so that the numbers of objects in the six consecutive blocks will form a partition of 251. Five of these blocks serve as x_1 through x_5 and the sixth block has "wasted" objects, since we want $x_1 + x_2 + x_3 + x_4 + x_5$ to be *at most* 251. Thus, you need to choose the 5 dividers among $251 + 5$ (objects and dividers) items.

Part II: (Variant of) Homework Question (20 points)

Note that even though the following question does not introduce any variation from its homework version, the question in the exam might.

4. [20 points]
n cars are standing at arbitrary points on a circle. Each car has a certain amount of fuel in it. Once a car a reaches another car b, a can take all the fuel that b has, and move forward. There is exactly enough fuel among all n cars to fuel any single one of them to make a full circle. Prove that there is a car that can move around the circle clockwise, taking fuel from other cars as it passes them, and run out of fuel just as it reaches its starting point.

SOLUTION: We prove by induction on n, the number of cars.

Base Case: $n = 1$. In this case, by the assumption in the problem statement, the car has enough petrol to make a complete circle.

Induction Hypothesis: Assume the result is true for n cars, for any $n \leq k$.

Induction Step: We prove this for $n = k + 1$. First, notice that there must be a car a that can reach its adjacent car b on the circle in the clockwise direction. Otherwise, if all cars have less petrol than what is required to reach their clockwise-next cars, then their total petrol would not be enough for a full circle. Now, any car (other than b) that reaches a can always use just a's petrol to reach b and add b's petrol to a's leftover petrol. Remove b from the circle, and transfer all its petrol to a. The new problem has k cars, thus by induction hypothesis, it has a car that can complete the full circle. Due to our reduction, it can also make a full circle in the original problem with $k+1$ cars.

Thus, the result holds by induction.

Part III: Longer Problems ($2 \times 25 = 50$ points)

Remember that in this part, you need to formally prove your answer for full credit. Be crisp, precise, and accurate!

5. [25 points]
A company Toca Cola(!) is holding a contest. They have issued n types of coupons, and everyone who collects at least one coupon of each type wins a prize. You get a coupon of a random type with each purchase of a Toca Cola. What is the expected number of Toca Colas you will need to buy in order to collect all the coupons? Your answer can be in the form of a summation, but its terms can only involve constants, variable(s) of the summation, and n.

SOLUTION: Let C be the random variable denoting the number of purchases required to collect at least one coupon of each type. Let us break this process in n stages. For $1 \leq i \leq n$, let C_i be the random variable denoting the number of purchases required to obtain a coupon of a new type (one that I do not already have) given that I have $i-1$ different types of coupons. Clearly, $C = \sum_{i=1}^{n} C_i$. Note that $C_1 = 1$ because I get my first coupon (of some type) with my first purchase.

Consider $i \geq 2$. After having obtained $i-1$ distinct types of coupons, the probability of getting a new type of coupon in the next purchase is $(n-i+1)/n$. Thus, C_i, which is the number of purchases required to get a coupon of a new type, is a geometric random variable with expectation $n/(n-i+1)$ (using the result from the class regarding the expectation of a geometric RV). Finally, using linearity of expectation,

$$\mathbb{E}[C] = \sum_{i=1}^{n} \mathbb{E}[C_i] = \sum_{i=1}^{n} \frac{n}{n-i+1}.$$

6. [25 points]
How many spanning trees (not necessarily minimum) does $K_{2,n}$ have? Recall that $K_{a,b}$ is the complete bipartite graph with a vertices on one side and b vertices on the other. You must give a closed form formula as a function of n for full credit.

SOLUTION: Let the two sets of vertices be $A = \{a_1, a_2\}$ and B where $|B| = n$.

Claim 1: In every spanning tree, at least one $b \in B$ must be connected to both a_1 and a_2.
Reason: Otherwise, a_1 and a_2 will be disconnected in the spanning tree, which is impossible.

Claim 2: In any spanning tree, no two distinct $b_1, b_2 \in B$ can both be connected to a_1 and a_2.
Reason: Otherwise, the spanning tree will have the cycle $a_1 - b_1 - a_2 - b_2 - a_1$, which is also impossible.

Hence, in every spanning tree, there must be exactly one $b \in B$ that is connected to both a_1 and a_2. There are n ways of choosing this vertex. Further, each of the remaining $n - 1$ vertices must be connected to either a_1 or a_2, otherwise the spanning tree would be disconnected. Hence, the number of ways of connecting them is 2^{n-1}. Note that connecting one vertex in B to both a_1 and a_2, and connecting every other vertex in B to exactly one of a_1 and a_2 indeed produces a connected graph without cycles, i.e., a spanning tree of $K_{2,n}$.

Thus, the total number of spanning trees is $n \cdot 2^{n-1}$.