15-251: Great Theoretical Ideas In Computer Science

Fall 2013

Midterm Test 2

Name:

Andrew ID:

Section:

INSTRUCTIONS:

- **Write your NAME, ANDREW ID, and SECTION above**.
- You may refer to your 3-by-5 index card, but no other notes. You may *not* use a calculator.
- Except for Part I (short answers), you must give clear and complete proofs. In your proofs, you may quote and use any results presented during lectures (unless specified otherwise); all other steps and results REQUIRE PROOFS. **Please write clearly**.

Problem	Points	Score	Problem	Points	Score
1	6		5	6	
2	6		6	20	
3	6		7	25	
4	6		8	25	
			Total	100	

Part I: Short answers ($5 \times 6 = 30$ points)

This part is to test your understanding of the basic concepts emphasized in class or recitations. You should be able to solve these problems using facts and concepts you know from memory, possibly supplemented with simple calculations. This part is meant to be relatively quick, but be sure to take enough time to read the questions carefully and calmly deduce your (hopefully correct) answers!

In questions that require explanation, show your work to (possibly) get partial credit. For multiple choice questions, simply put a tick mark on the correct answer(s). **Note that some questions may have multiple correct answers. You get full credit if you tick all (and only) the correct answers, and no credit otherwise.**

1. [6 points]
What is the relation between $f(n) = 2^{\log^* n}$ and $g(n) = n$? (Multiple answers may be correct.)

a) $f(n) \in O(g(n))$
b) $f(n) \in \Theta(g(n))$
c) $f(n) \in \Omega(g(n))$
d) None of the above

2. [6 points]
Consider the following problem: Given an undirected graph G and a number k, determine whether all *maximal* (not necessarily maximum cardinality) matchings of G are of size *at most* k. Which of the following statements are known to be true about this problem? (Assume $P \neq NP$. Multiple answers may be correct.) [Hint: A maximum cardinality matching is also a maximal matching.]

a) It is in P.
b) It is in NP.
c) It is NP-complete.
d) It is outside NP.

3. [6 points]
Recall the CleverVoting system from Homework 8 where each voter gives a fixed decreasing sequence of points ($\alpha_1 > \alpha_2 > \ldots > \alpha_k$) to candidates in his preference list in that order (instead of simply giving $k-1, k-2, \ldots, 0$ points as in the Borda count). Which of the following properties would every such voting rule (independent of the choice of αs) satisfy? (Multiple answers may be correct.)

a) Unanimity (if all voters put the same candidate first, it must be the winner)
b) Majority consistency (if a majority of the voters put the same candidate first, it must be the winner)
c) Condorcet consistency
d) None of the above

4. [6 points]
Consider the ski rental algorithm that rents for $3B$ days and then buys (buying costs $\$B$, renting costs $\$1$). Of the following four values of c, what is the smallest one such that this algorithm is c-competitive?

a) 2 b) 3 c) 4 d) 5

5. [6 points]
Consider the problem "CLIQUE" of finding the maximum size of any clique in a given graph, and the algorithm GREEDY that starts from the highest degree vertex and successively adds its neighbours as long as the added vertices remain a clique. Give an example of an undirected graph (possibly not connected) where GREEDY is NOT a $(1/2)$-approximation for CLIQUE.

Part II: (Variant of) Homework Question (20 points)

6. [20 points]
Let $G = (V, E)$ be a directed graph such that for every two vertices $u, w \in V$, exactly one of the two directed edges $u \to w$ and $w \to u$ is in E. Prove that G has a Hamiltonian path.

Part III: Longer Problems ($2 \times 25 = 50$ points)

Remember that in this part, you need to formally prove your answer. Be crisp, precise, and accurate!

7. [25 points]
Suppose you have n indivisible objects (say a keyboard, a mouse, a monitor etc.) and two players. Each player i has some value $v_{ij} \in \mathbb{R}_{\geq 0}$ for each object j. Prove that checking whether there exists an envy-free division of these objects between the two players is NP-complete. Use PARTITION as your NP-complete problem. [Hint: Create identical valuations!]

8. [25 points]
Consider the following SET-COVER problem.

SET-COVER: Given a set of elements $U = \{e_1, \dots, e_n\}$ and subsets $S_1, \dots, S_t \subseteq U$ such that $\bigcup_{i=1}^{t} S_i = U$, find the minimum number of subsets whose union is U. That is, find the minimum number of subsets that need to be picked to "cover all elements".

Notice that this is different from the MAX-COVER problem from Homework 8; while MAX-COVER requires covering the *maximum number* of elements with a *given number of subsets*, SET-COVER asks to pick the *minimum number* of subsets to *cover all elements*. Thus, SET-COVER is a minimization problem. Consider the following GREEDY algorithm (same as the greedy algorithm for MAX-COVER). First pick the subset that has the maximum number of elements. Then, iteratively pick the subset that has the maximum number of yet uncovered elements.

Now, consider the following example. We have a class of students where there are 2^{i-1} boys and 2^{i-1} girls of each age $1 \le i \le k$. For each i, let A_i be the set of people of age i (there are 2^i of them). Let B be the set of boys, and G be the set of girls, so $|B| = |G| = \sum_{i=1}^{k} 2^{i-1}$. Consider the instance of SET-COVER where U is the set of all students in the class, and there are $k+2$ subsets available, namely $A_1, \dots, A_k$, B and G. Note that in this instance $n = 2\sum_{i=1}^{k} 2^{i-1}$.

Use this example to show that if GREEDY is an $f(n)$-approximation to SET-COVER, then $f(n) \in \Omega(\log n)$.

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