15-251: Great Theoretical Ideas In Computer Science

Recitation 6 Solutions

Coloring

(a) How many ways are there to k-color a tree with v vertices?

$k \cdot (k-1)^{v-1}$.

Color one node (k choices). Then, color in nodes that are neighbors of an already-colored node, one at a time ($k-1$ choices at each step, since exactly 1 neighbor is already colored). This will color all nodes, since trees are connected, and no uncolored node will border two colored nodes at any point because it is a tree and this would imply a cycle, which trees don't have.

Graph-iti, Probably

(a) Let $G = (V, E)$ be a connected graph and randomly choose a subgraph $G' = (V', E')$ of G such that each vertex in V is in V' with probability p, and each edge is in E' if and only if both of its incident vertices are in V'.

Construct a set of vertices H as follows: for each edge e in E', randomly choose one of its incident vertices to not be in H. H is the set of vertices that were never chosen. Find a value of p that $|H| \geq \frac{n^2}{4m}$ where $n = |V|$ and $m = |E|$, respectively.

First, note that every edge in E' will remove at most 1 node from V' to get to H.
So $E[|H|] \geq E[|V'| - |E'|]$. By linearity of expectation,
$E[|H|] \geq E[|V'|] - E[|E'|]$.

Note that $E[|V'|]$ is just pn (number of vertices in the original graph times probability of choosing a vertex), and $E[|E'|]$ is just $p^2 m$. So we have
$E[|H|] \geq pn - p^2 m$. If we let $p = \frac{n}{2m}$, then we get exactly our equation to be proven:
$E[|H|] \geq \frac{n^2}{2m} - \frac{n^2}{4m} = \frac{n^2}{4m}$.

(Note that $\frac{n}{2m} < 1$ because the graph is connected, meaning that there are at least $n-1$ edges.)

Hypercubes and Ultracubes

(a) An n-cube is a cube in n dimensions. A cube in one dimension is a line segment; in two dimensions, it's a square, in three, a normal cube, and in general, to go to the next dimension, a copy of the cube is made and all corresponding vertices are connected. If we consider the cube to be composed of the vertices and edges only, show that every n-cube has a Hamiltonian cycle.

Proof by induction.
Base case $n = 1$ is trivial.
Assume there exists a Hamiltonian cycle on a k-cube. To prove that a cycle for a $k+1$-cube exists:,
Note that a $k+1$-cube is constructed by taking two copies of a k-cube and connecting the corresponding vertices.
Take a Hamiltonian cycle for a k-cube and delete the last edge from the cycle. Call this path P, and say it starts at a and ends at b. Let the reverse of this path be P', starting at b and ending at a. Follow P on one copy of the k-cube and P' on the other copy (going from b' to a'). Since there is an edge between a and a' and one between b and b', we can start at a, follow P to b through every point in cube 1, go to b', follow P' to a' through every point in cube 2, then back to a.

A Very Average Tree

(a) What is the average number of spanning trees for simple labeled graphs with n vertices?

Let $s(G)$ be the number of spanning trees of a graph G, and let $g(T)$ be the number of graphs having tree T as a spanning tree. Then
$\sum_G s(G) = \sum_T g(T)$ (you can count the number of spanning trees of a graph and sum across graphs, or count the number of graphs for a tree and sum across trees). There are $2^{\binom{n}{2}}$ possible graphs on n vertices (because there are $\binom{n}{2}$ possible edges), so

$$\frac{\sum_G s(G)}{2^{\binom{n}{2}}} = \frac{\sum_T g(T)}{2^{\binom{n}{2}}} = \text{our average.}$$

Note that $|E(T)| = n-1$ (the number of edges in the tree). Note that to count the graphs that have T as a spanning tree, we simply count the number of ways to pick edges not in T: $2^{\binom{n}{2}-(n-1)} = g(T)$ for all T. Since by Cayley's formula, there are n^{n-2} trees on n vertices, the average is

$$\frac{n^{n-2} \cdot 2^{\binom{n}{2}-(n-1)}}{2^{\binom{n}{2}}} = \frac{n^{n-2}}{2^{n-1}}.$$

A Little Time Complexity

Assume all functions are from the positive integers to the positive integers.

(a) Prove or disprove: "For all f, g, at least one of $f \in O(g)$, $g \in O(f)$ is true."

False. Counterexample:

$$f(n) = \begin{cases} n!, & n \text{ is even} \\ (n-1)!, & n \text{ is odd} \end{cases}, \quad g(n) = \begin{cases} (n-1)!, & \text{is even} \\ n!, & n \text{ is odd} \end{cases}$$

(b) Prove or disprove: "For all f, g, $f(n) \in \Theta(g(n))$ implies that $f(n)^3 \in \Omega(g(n)^2)$.

True.
$f(n) \in \Theta(g(n)) \Rightarrow f(n) \in \Omega(g(n))$
So there exist c, N such that $f(n) \geq c \cdot g(n)$ for all $n \geq N$.
Cubing both sides, $f(n)^3 \geq c^3 g(n)^3 \geq c^3 g(n)^2$ since $g(n)$ is a positive integer.
So there exist c', N' such that $f(n)^3 \in \Omega(g(n)^2)$: let $c' = c^3$ and $N' = N$.

(c) Prove or disprove: "For all f, g, $f(n) \in \Theta(g(n))$ implies that $f(n) \notin \Theta(g(n)^2)$

False. Counterexample: $f(n) = g(n) = 1$.
$g(n)^2 = 1$, therefore $f(n) \in \Theta(g(n)^2)$.

(d) Prove or disprove: "For all f, if $f(n) \in \Omega(n^k)$ for all k, then $f(n) \in \Omega(2^n)$.

False. Counterexample: $f(n) = \lfloor 1.5^n \rfloor$. This grows faster than any polynomial function but slower than 2^n.