15-251: Great Theoretical Ideas In Computer Science

Recitation 4 Solutions

Expected Value

(a) You roll 2 3-sided die.

1. Let S be the sum of the two die. What is $E(S)$?
2. Let D be the difference of the two die. What is $E(D)$?
3. Let M be twice the maximum of the two die. What is $E(M)$?

Possibilities are $(1,1)$, $(1,2)$, $(1,3)$, $(2,1)$, $(2,2)$, $(2,3)$, $(3,1)$, $(3,2)$, and $(3,3)$.

Part 1:
Possible sums are 2, 3, 4, 5, and 6, with corresponding probabilities $1/9$, $2/9$, $3/9$, $2/9$, $1/9$.
Then, $E(S) = \sum_{i=2}^{6} iP(S = i) = 2(1/9) + 3(2/9) + 4(3/9) + 5(2/9) + 6(1/9) = 36/9$.

Part 2:
Possible differences are 0, 1, 2, with corresponding probabilities $3/9$, $4/9$, $2/9$.
Then, $E(D) = \sum_{i=0}^{2} iP(D = i) = 0(3/9) + 1(4/9) + 2(2/9) = 8/9$.

Part 3:
Can do as before, but note that $M = S + D$.
Thus $E(M) = E(S + D) = E(S) + E(D) = 36/9 + 8/9 = 44/9$.

(b) You put 6 pairs of socks in the dryer, where each pair is distinct. However, only 9 socks remain when you remove your laundry. What is the expected number of pairs that we now have?

Consider the first pair. There are $\binom{12}{9} = 220$ total possibilities for the 9 socks left, and $\binom{10}{7} = 120$ possible sets of remaning socks that include the first pair (include the pair and then choose 7 from remaining 10 socks). Thus the probability that the first pair remains is $\frac{120}{220} = \frac{6}{11}$.

Similarly, each pair has a probability $\frac{6}{11}$ of remaining.

Let X_i be the indicator random variable that is 1 if pair i remains and 0 otherwise. Thus we wish to compute $E(\sum_{i=1}^{6} X_i)$.

By linearity of expectation, this is $\sum_{i=1}^{6} E(X_i)$. Since each X_i is an indicator random variable, we have from before that $E(X_i) = \frac{6}{11}$.

Thus the expected number of pairs remaining is $6(\frac{6}{11}) = \frac{36}{11}$.

(c) m people enter the elevator on the ground floor of a building with n additional floors. Each of them independently selects a floor, each with equal probability. What is the expected number of floors the elevator will stop at?

For each i from 1 to n, let $X_i = 1$ if we stop at the ith floor and 0 otherwise. Then, we wish to determine $E(\sum_{i=1}^{n} X_i)$.

By linearity of expectation, we have $E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i)$. Each X_i is just an indicator random variable, so $E(X_i) = P(X_i = 1)$.

Then, the probability that we don't go to floor i is $(\frac{n-1}{n})^m$, so $P(X_i = 1) = 1 - (\frac{n-1}{n})^m$.

Thus the expected number of floors the elevator stops at is $n(1 - (\frac{n-1}{n})^m)$.

Graphs

(a) A k-regular graph is one where every vertex has degree k. How many undirected 3-regular graphs are there on 7 vertices?

None. The sum of the degrees is $3(7) = 21$, which is odd. By the handshaking theorem, however, the sum of the degrees is twice the number of edges, and therefore must be even.

(b) n lines are drawn in the plane. None of them are parallel, and no three of them intersect at a common point. A large circle is drawn around all intersections created by these n lines (strictly around - the circle does not pass directly through any intersections). How many regions are there inside this circle?

We consider the planar graph contained in the circle (including the circle), where vertices are the points of intersection between two lines or the circle and a line.

There are $\binom{n}{2} + 2n$ vertices in this graph - $\binom{n}{2}$ because there is one for each pair of lines, and $2n$ because each line intersects the circle twice.

Each vertex resulting from the intersection of two lines has degree 4, and the outer vertices (which resulted from the intersection between the circle and a line) have degree 3. Thus the sum of the degrees is $4\binom{n}{2}+6n$, so the number of edges is $2\binom{n}{2}+3n$ by the handshaking lemma.

Now, since this graph is planar, we use Euler's formula: $V - E + F = 2$.
This gives us $\binom{n}{2} + 2n - (2\binom{n}{2} + 3n) + F = 2$, so $F = 2 + \binom{n}{2} + n$.

This isn't quite what we want, as we don't want to include the outer face. Thus the number of regions is $\binom{n}{2} + n + 1 = \binom{n+1}{2} + 1$.

(c) The complement of a graph $G = (V, E)$ is $\bar{G} = (V, E')$ where $E' = \{e | e \notin E\}$. Prove that the complement of a planar graph on at least 11 vertices is not planar.

Let $G = (V, E)$ be a planar graph with $|V| \geq 11$. Let $n = |V|$ and $m = |E|$. From class, $m \leq 3n - 6$. Now, $\bar{G}$ has $\binom{n}{2} - m$ edges and n vertices.

Suppose for sake of contradiction that $\bar{G}$ is planar. Thus we have

$$\binom{n}{2} - m \leq 3n - 6$$

$$\implies \frac{n(n-1)}{2} - (3n-6) \leq 3n - 6$$

$$\implies n^2 - n \leq 12n - 24$$

$$\implies n^2 - 13n + 24 \leq 0$$

$$\implies \frac{13 - \sqrt{(169-96)}}{2} \leq n \leq \frac{13 + \sqrt{(169-96)}}{2}$$

$$\implies n < \frac{13 + \sqrt{(81)}}{2} = 11$$

Contradiction.