15-251: Great Theoretical Ideas In Computer Science

Recitation 2 Solutions

Bad Counting

What is wrong with the following counting arguments?

(a) How many orderings of the letters A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P are there such that we cannot obtain any of the words BAD, DEAF, APE by crossing out some letters?

Solution: Since A is in all three words, we choose the spot for A first.
We know the B has to go after the A, otherwise we could have BAD.
The D also has to go after A, or we could have DEAF.
The E has to go before either the P or the A, or we could have APE.
So in the case where the E goes before the A, we treat A,B,D,E as a group (then put the A in the first spot and permute the others), and place the others randomly to get $3!\binom{16}{4}12!$. In the case where E goes before the P, we treat A,B,D as a group and E, P as a group and choose the spots for all 5 of them, then distribute them to get $\binom{16}{5}\binom{5}{2}2!11!$.
Then, to get the number of orderings, we take the sum of these two quantities.

There is quite a lot wrong with this proof, but the most glaring flaw is that it only takes care of some of the conditions (and enforces some that do not necessarily always need to be true). For instance, If the E goes before the D, the A doesnt need to be before the B. We can use inclusion-exclusion (on combinations of the words being considered).
$16!$ is the total number of permutations of the letters. We can view each of the words as a problem of choosing the three (or four) spaces for the r letters. Then, there are $(16-r)!$ ways to place the remaining letters. So, for BAD, we have $\binom{16}{3} \times 13!$; for DEAF, we have $\binom{16}{4} \times 12!$; and for APE, we have $\binom{16}{3} \times 13!$.
The only possible overlap is between BAD and APE; the others explicitly have letters in different orders. The only possible ways to make an overlap between BAD and APE are BADPE, BAPDE, and BAPED, and using the same idea, there are $3 \times \binom{16}{5} \times 11!$ ways to do that.
Putting it all together with inclusion-exclusion, we get:

$$16! - \binom{16}{3} \times 13! - \binom{16}{4} \times 12! - \binom{16}{3} \times 13! + 3\binom{16}{5} \times 11! \quad (1)$$

(b) 20 people are sitting at a round table. How many ways are there of choosing 3 people from them so that no two of the chosen are neighbors?

Solution: There are 20 ways of choosing the first person, 17 ways of choosing the second person, since he can't sit next to or on top of the first person, and there are 14 ways of choosing the third person. The order we choose these people however does not matter, so we divide by $3!$ and $\frac{20\times 17\times 14}{3!}$.

The argument doesn't consider the situation where the first and second people are "almost neighbors," meaning that there is only one empty space between them. When this happens there are 15 possible ways for the third person. Order still does not matter so we have:

$$\frac{20 \times (2 \times 15 + 15 \times 14)}{3!} = 800 \tag{2}$$

Another way to get at the same answer is via the inclusion exclusion principle. First there are $\binom{20}{3}$ ways to choose three seats, with no restriction. Then there are 20×18 ways to choose one pair of seats and then one other seat. This is the number of ways in which at least two people are sitting next to each other. Visually you can think of it like this: pick one seat, place one person there. Sit the second person immediately to the left of this person. Then there are 18 places to place the third person. This gives 20×18 ways in which at least two people are next to each other.
However, notice that we double-counted the configurations where all three of them are next to each other. To see why, note that picking the third person to be immediately to the left of the second person is the same as moving the first person one seat to the left and picking the third person to sit immediately to the right of the first person. Thus we have to add back the number of ways we can have all three of them sitting together. There are 20 ways to do this. Thus the answer is:

$$\binom{20}{3} - 20 \times 18 + 20 = 800 \tag{3}$$

251 Seating, Continued

(a) In lecture, we went over the number of ways to seat 151 boys and 100 girls in a row such that no two girls sit next to each other. Now, what if we add the restriction that no more than two boys can sit between any two girls?

The students are considered to be indistinguishable here. We consider three cases.
The first case is when there are boys on both ends of the row. Then, there are 101 spots in which to place 151 boys; we know each spot must have at least one boy, so that the problem reduces to finding the number of ways to place $151 - 101 = 50$ leftover boys in the 101 spots so that none of those left over boys share the same spot (this way, not spot has more than 2 boys). This gives $\binom{101}{50}$ ways to do this.
Now consider the case where there is a boy is on only on end of the row. Then, there are 100 spots in which to place 151 boys; we know each spot must have at least one boy, so that the problem reduces to finding the number of ways to place $151 - 100 = 51$ leftover boys in the 100 spots so that none of those left over boys share the same spot (this way, not spot has more than 2 boys). This gives $\binom{100}{51}$ ways. This case is considered twice (one for both ends of the row), so that there is a total of $2\binom{100}{51}$ ways to do this.
The final case is where no boys are on either end of the row. The case is similar to the previous two, except that there are only 99 spots to place the boys. Following similar reasoning, this gives $\binom{99}{52}$ ways to place the boys.
Thus, there are $\binom{101}{50} + 2\binom{100}{51} + \binom{99}{52}$ ways to seat students so.
If the students are treated to be distinguishable, we multiply the above result by $151!100!$ to count the permutations of the boys and the permutations of the girls.

(b) What if we remove that restriction, but the students are now sitting in a circle? (the restriction that no two girls sit next to each other remains)

We consider the students to be indistinguishable here. Firstly, assume that rotations are considered to be different seatings. This is similar to the first problem, but different in that now there are 151 bins in which to distribute 100 girls. This solves to $\binom{151}{100}$ ways to seat the students. Now, because the number of boys is prime and the number of girls present is relatively prime to the number of boys present, no matter how you rotate the positions of the girls (there are 151 ways to rotate the girls since there are 151 spaces for the 100 girls to occupy), the rotations you get will be unique (modulo 151, meaning if a seating arrangement A is rotated by m bins to the right, and the same A is rotate by n bins to the right, for $m \neq n$, $0 \leq m, n < 251$, then the resulting seatings are different).
Say by contradiction that there were two such m and n. WLOG, we can assume $m = 0$. Then if $n < 251$, then rotating by any multiple of n should give the same seating as A. Since n is relatively prime to m, we can find a multiple of n that is congruent to $1 \mod 251$, so that rotating A once gives the same seating as not rotating at all. This is clearly contradictory, as there are less girls than there are bins, so that there must be some consecutive 2 bins where one is occupied and the other is not, so that by rotating it once, the seating cannot be the exact same.
Therefore, if we are to consider each rotation as the same, we have overcounted by 151, giving a final answer of $\frac{1}{151}\binom{151}{100}$ ways.
If the students are distinguishable, as before, we multiply the above result by $151!100!$.