CMU 15-251
Online Algorithms

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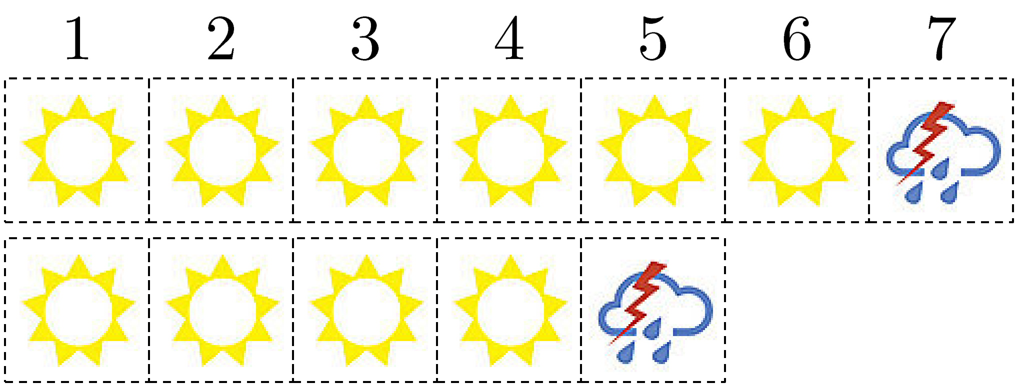
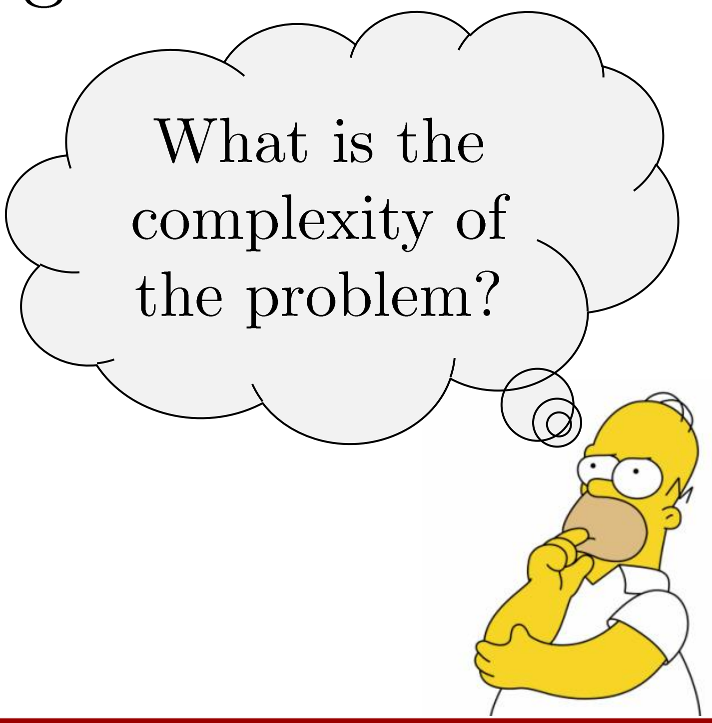
Ski rental

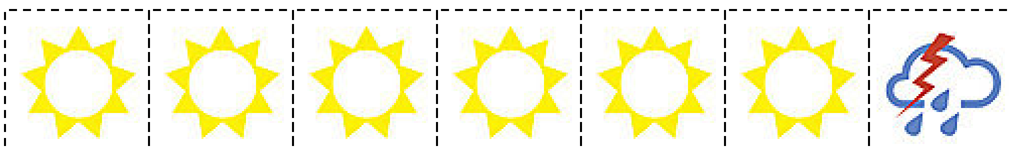
- You are on a ski vacation; you can buy skis for $\$B$ or rent for $\$1/\text{day}$
- You're very spoiled: You'll go home when it's not sunny
- Rent or buy when $B = 5$?



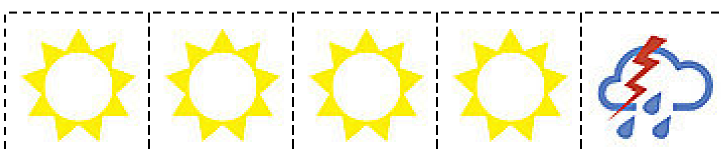


Ski rental

- Now assume you don't know in advance how many days of sunshine there are
- Every day of sunshine you need to decide whether to rent or buy
- **Algorithm:** Rent for B days, then buy



$\Rightarrow \$10$



$\Rightarrow \$4$

Ski rental

Vote: Assume $B \geq 8$. How bad can the "rent B days, then buy" algorithm be compared to the optimal solution in the worst case?

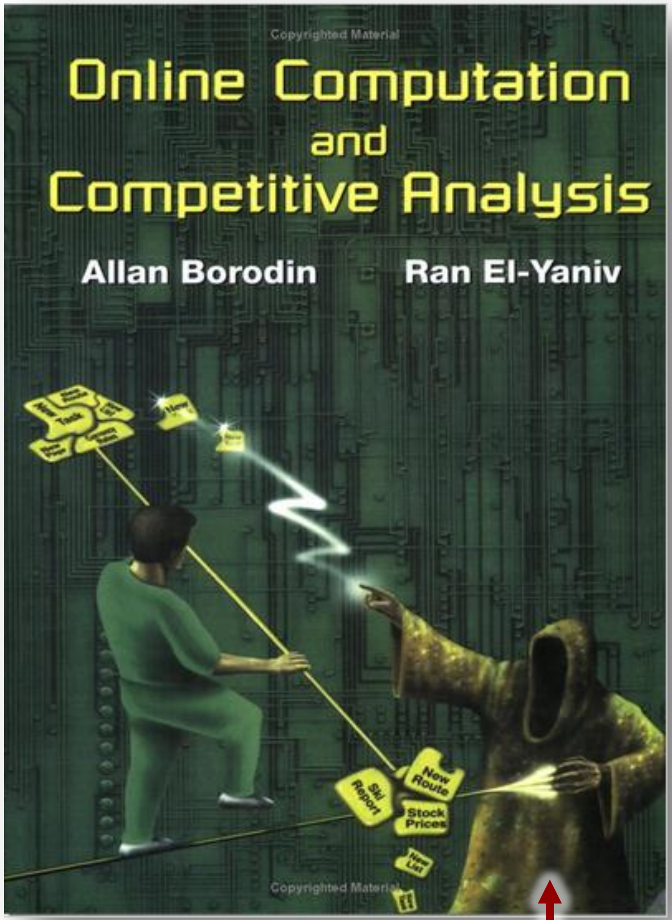
1. $ALG(I) = 2 \cdot OPT(I)$
2. $ALG(I) = 3 \cdot OPT(I)$
3. $ALG(I) = \frac{B}{2} \cdot OPT(I)$
4. $ALG(I) = B \cdot OPT(I)$

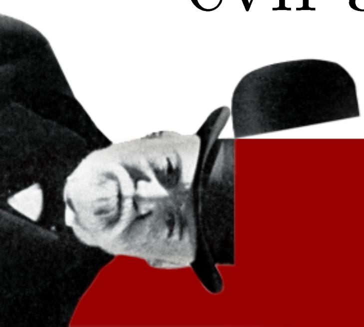
Competitive ratio

- For a minimization problem and $c > 1$, ALG is a c-competitive algorithm if for every instance I, $ALG(I) \leq c \cdot OPT(I)$
- For a maximization problem and $c < 1$, ALG is a c-competitive algorithm if for every instance I, $ALG(I) \geq c \cdot OPT(I)$
- The difference from approximation algorithms is that here ALG is online, whereas $OPT(I)$ is the optimal offline solution

Ski rental, revisited

- Our ski-rental algorithm is 2-competitive
- Renting for $B - 1$ days is $\left(\frac{2B-1}{B}\right)$-competitive
- We prove that no online algorithm can do better by constructing an evil adversary



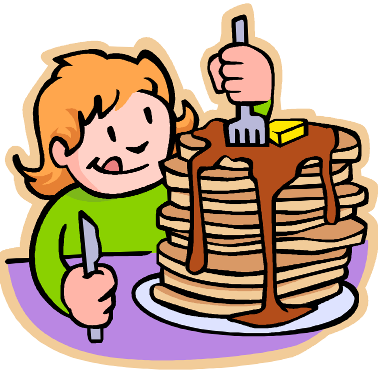


Ski rental, revisited

- **Theorem:** No online algorithm for the ski rental problem is α-competitive for $\alpha < \frac{2B-1}{B}$
- **Proof:**
  - Alg is defined by renting for K days and buying on day $K + 1$
  - Evil adversary makes it rain on day $K + 2$
  - $K \geq B$: $OPT(I) = B, ALG(I) = K + B \geq 2B$
  - $K \leq B - 2$: $OPT(I) = K + 1$,
    $$ALG(I) = K + B \geq 2K + 2 \blacksquare$$

Pancakes, revisited

$\approx$



Competitive analysis | Pancakes

"The Bth ski number is $\frac{2B-1}{B}$"

Ski rental, revisited

Proving lower bounds for online algorithms is much easier than for approximation algorithms!

Paging

- Hard drive holds N pages, memory holds k pages
- When a page of the hard drive is needed, it is brought into the memory
- If it's already in the memory, we have a hit, otherwise we have a miss
- If the memory is full, we may need to evict a page
- Paging algorithm tries to minimize misses

Paging

Memory			Request sequence
1	2	3	4
4	2	3	4 1
4	1	3	4 1 3
4	1	3	4 1 3 2
2	1	3	4 1 3 2 4

Paging

Memory			Request sequence
1	2	3	4
1	4	3	4 1
1	4	3	4 1 3
1	4	3	4 1 3 2
2	4	3	4 1 3 2 4

Paging

- Four online paging algorithms (start with $1, \ldots, k$) in memory
- LRU (least recently used)
- LFU (least frequently used)
- FIFO (first in first out): memory works like a queue; evict the page at the head and enqueue the new page
- LIFO (last in first out): memory works like a stack; evict top, push new page

Example: LIFO

Memory	Request sequence
1 2 3	4
1 2 4	4 3
1 2 3	4 3 4
1 2 4	4 3 4 3
1 2 3	4 3 4 3 4

Paging

- Vote: What is the smallest α for which LIFO is α-competitive?
  1. $\alpha = 2$
  2. $\alpha = k$ (size of memory)
  3. $\alpha = N$ (number of pages)
  4. $\alpha = \infty$ (can't be bounded with these parameters)

Paging

- Vote: What is the smallest α for which LFU is α-competitive?
  1. $\alpha = 2$
  2. $\alpha = k$
  3. $\alpha = N$
  4. $\alpha = \infty$

Paging

- **Theorem:** LRU is k-competitive
- **Proof:**
  - We divide the request sequence into phases; phase 1 starts at the first page request; each phase is the longest possible with at most k requests for distinct pages
  - Example with $k = 3$:

Phase 1	Phase 2	Phase 3
4 1 2 1	5 3 4 5	1 2 3

Paging

- **Theorem:** LRU is k-competitive
- **Proof (continued):**
  - Denote $m =$#stages, and by p_j^i the jth distinct page in phase i
  - Pages $p_1^i, \dots, p_k^i, p_1^{i+1}$ are all distinct
  - If OPT hasn't missed on pages $p_2^i, \dots, p_k^i$, it will miss on p_1^{i+1}, i.e., it misses at least once for every new phase (including phase 1)
    $\Rightarrow OPT \geq m$

Paging

- **Theorem:** LRU is k-competitive
- **Proof (continued):**
  - LRU misses at most once on each distinct page in a phase
  - Therefore, $ALG \leq km$ ■

4	1	2	5
5	1	2	5 3
5	1	3	5 3 4
5	4	3	5 3 4 5

Phase 2 of the example on slide 15

Paging

- **Theorem:** FIFO is k-competitive
- **Proof:** Essentially the same ■
- **Theorem:** No online alg for the paging problem is α-competitive for $\alpha < k$

Paging

- **Proof:**
  - At each step the evil adversary requests the missing page in $\{1, \ldots, k+1\} \Rightarrow$ miss every time

1	2	3	4
4	2	3	4 1
4	2	1	4 1 3
4	3	1	4 1 3 2
4	3	2	4 1 3 2 1

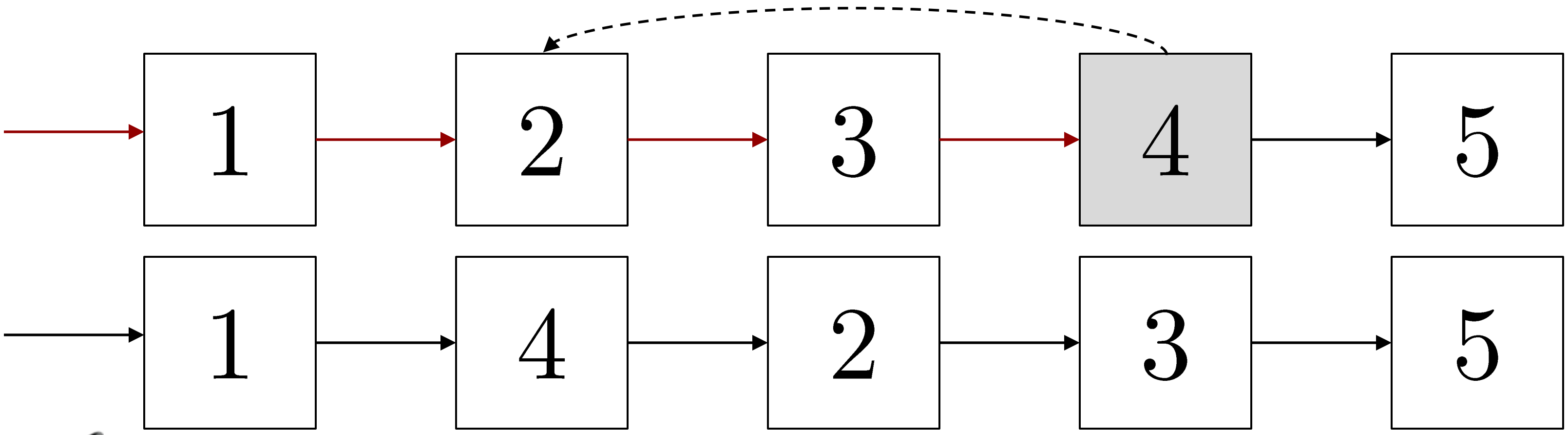
Paging

- **Proof:**
  - If OPT evicts a page, it will take at least k requests to miss again ■

1	2	3	4
1	4	3	4 1
1	4	3	4 1 3
1	4	3	4 1 3 2
1	4	2	4 1 3 2 1

List update

- Linked list of length n
- Each request asks for an element; traverse links to element; pay 1 for each such link
- Allowed to move requested element up the list for free



List update

- Three list update algorithms
- Transpose: Move requested element one position up (if it's not first)
- Move to front: Move requested element to the head of the list
- Frequency counter: Keep track of how many times each element was requested; move requested element past elements that were requested less frequently

List update

- Vote: Which algorithm is α-competitive for a constant α?
  1. Transpose
  2. Move to front
  3. Frequency counter

What we have learned

- Definitions:
  - Competitive algorithm
  - Ski rental, paging, list update problems
- Algorithms:
  - Competitive algs for ski rental, paging
- Principles:
  - Evil adversary!