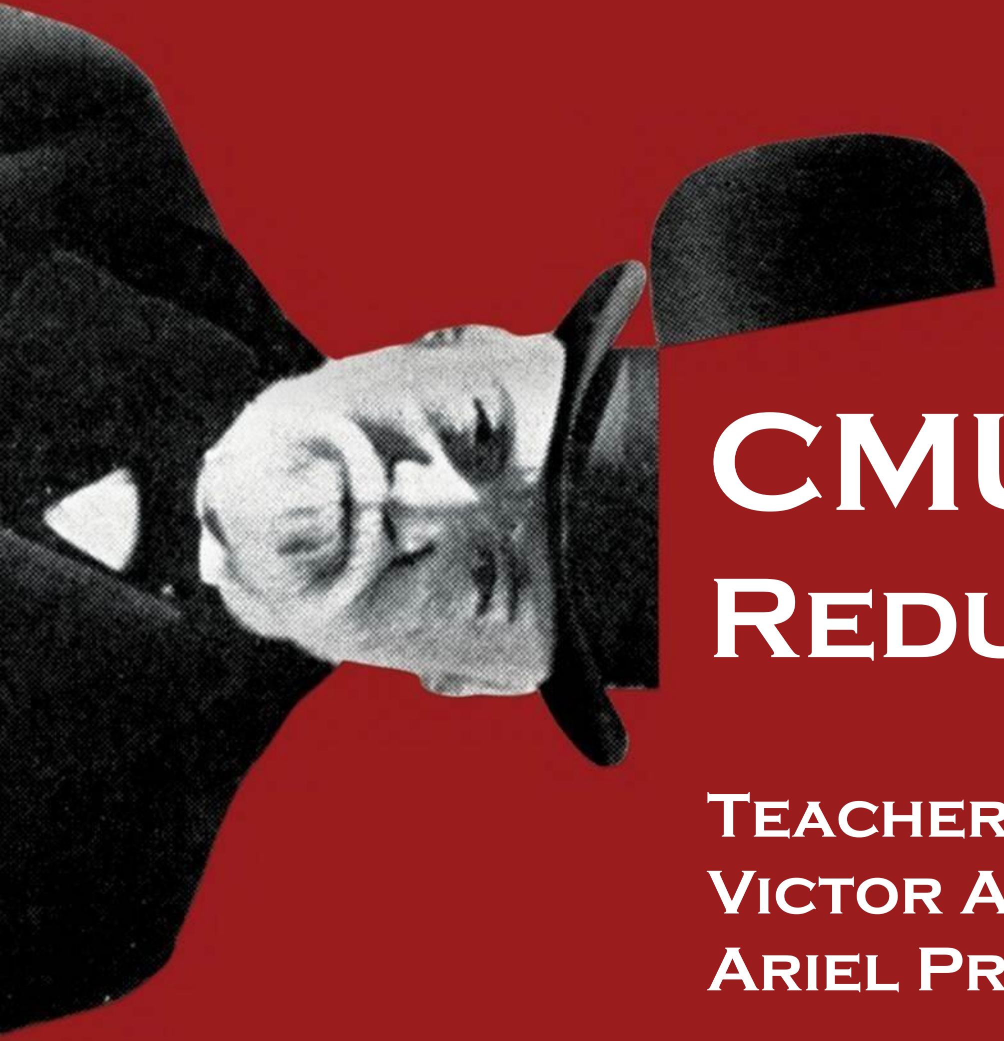


CMU 15-251
Reductions

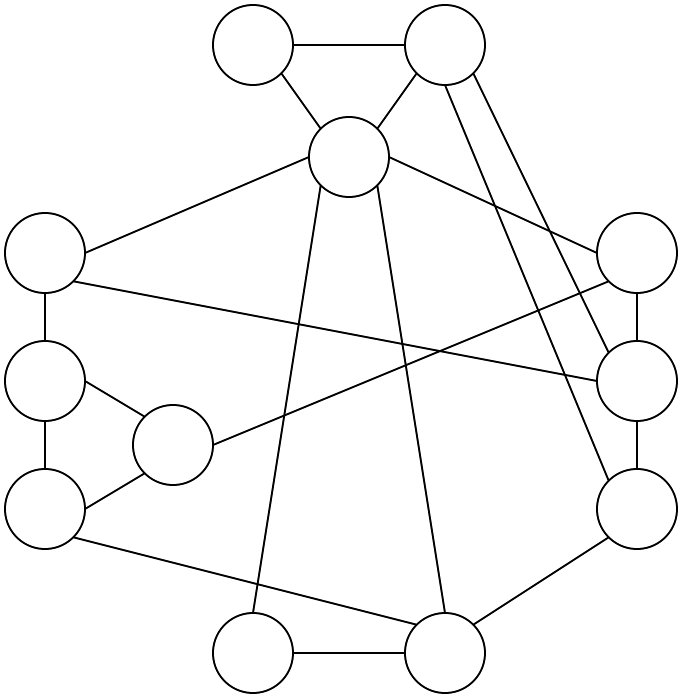
Teachers:
Victor Adamchik
Ariel Procaccia (this time)

k-COLORING

- Reminder: a k-coloring of a graph satisfies:
  - Each node has a color
  - There are at most k different colors
  - Every two nodes connected by an edge have different colors
- A graph is k-colorable iff it has a k-coloring

2-coloring

- Is this graph 2-colorable?



2-COLORING

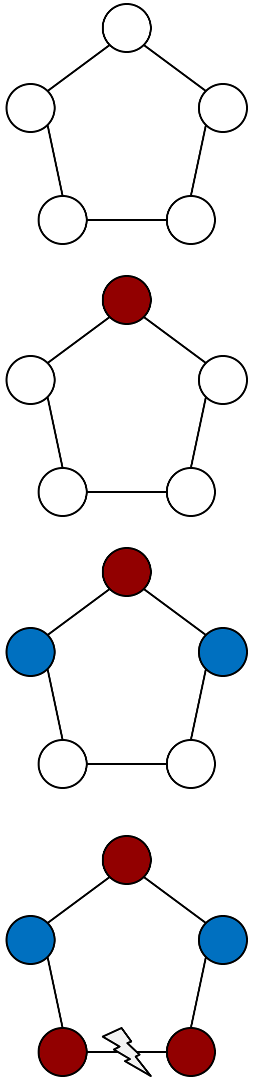
- Given a graph G, how can we decide if it is 2-colorable?
- Enumerate all possible 2^n colorings to look for a valid one…
- OK, but how can we efficiently decide if G is 2-colorable?
  - In polynomial time in the number of vertices n

2-COLORING

- Vote: $G = (V, E)$ is 2-colorable iff:
  1. G has a Hamiltonian cycle
  2. G has an Eulerian cycle
  3. G has no odd cycles
  4. G has no even cycles
  5. $|E| \leq |V| + 1$
  6. $|E| \geq |V| + 1$

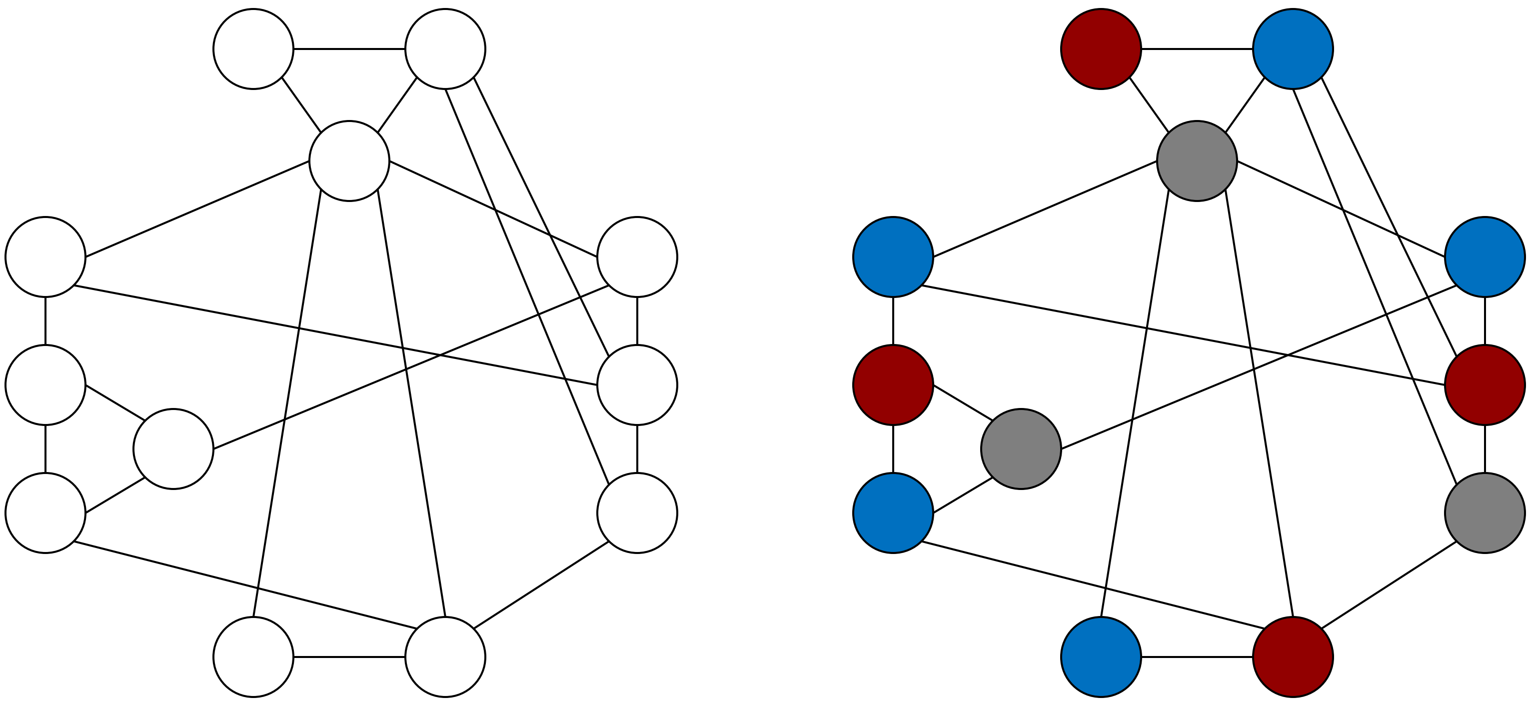
2-COLORING

- Algorithm:
  - Choose an arbitrary node, color it red and its neighbors blue
  - Color the uncolored neighbors of the blue vertices red, etc.
  - If G is not connected, repeat for every component



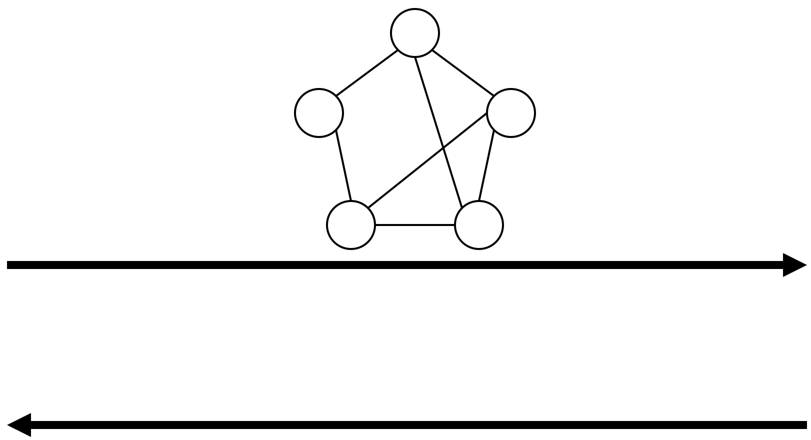
3-COLORING

- Is this graph 3-colorable?



3-colorability oracles

- We can decide 3-colorability by trying all 3^n possible colorings
- Let's say we can ask an oracle…

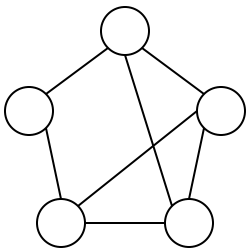


NO / YES

3-colorability decision oracle

3-colorability oracles

- How do we turn a decision oracle into a search oracle?

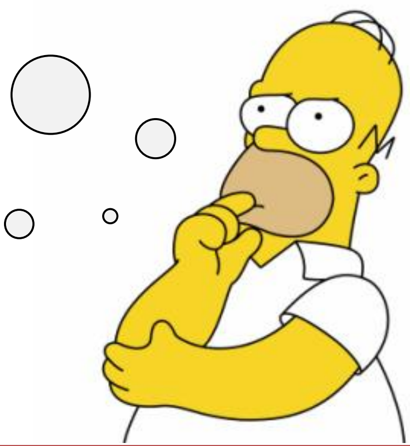


NO / YES, here's how

3-colorability search oracle

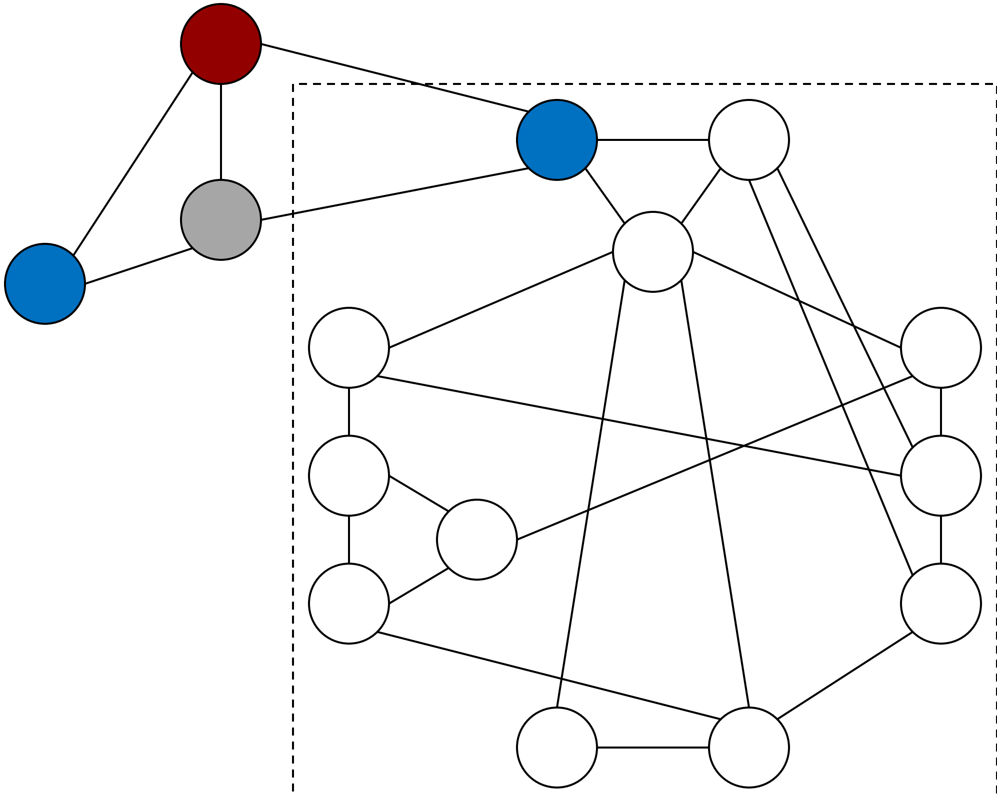
3-COLORABILITY ORACLES

What if I gave the oracle partial colorings of G? For each partial coloring of G, I could pick an uncolored node and try different colors on it until the oracle says "YES". I would then have a larger partial coloring



The oracle doesn't accept partial colorings!

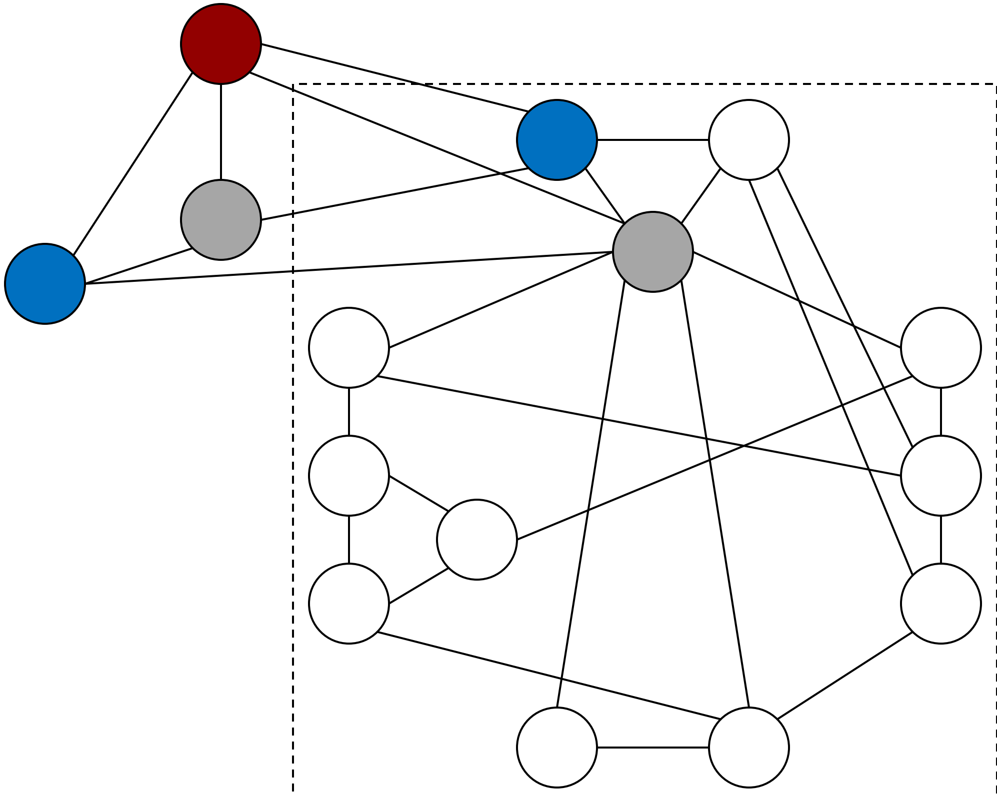
3-colorability oracles



Given:
3-colorability decision oracle

YES

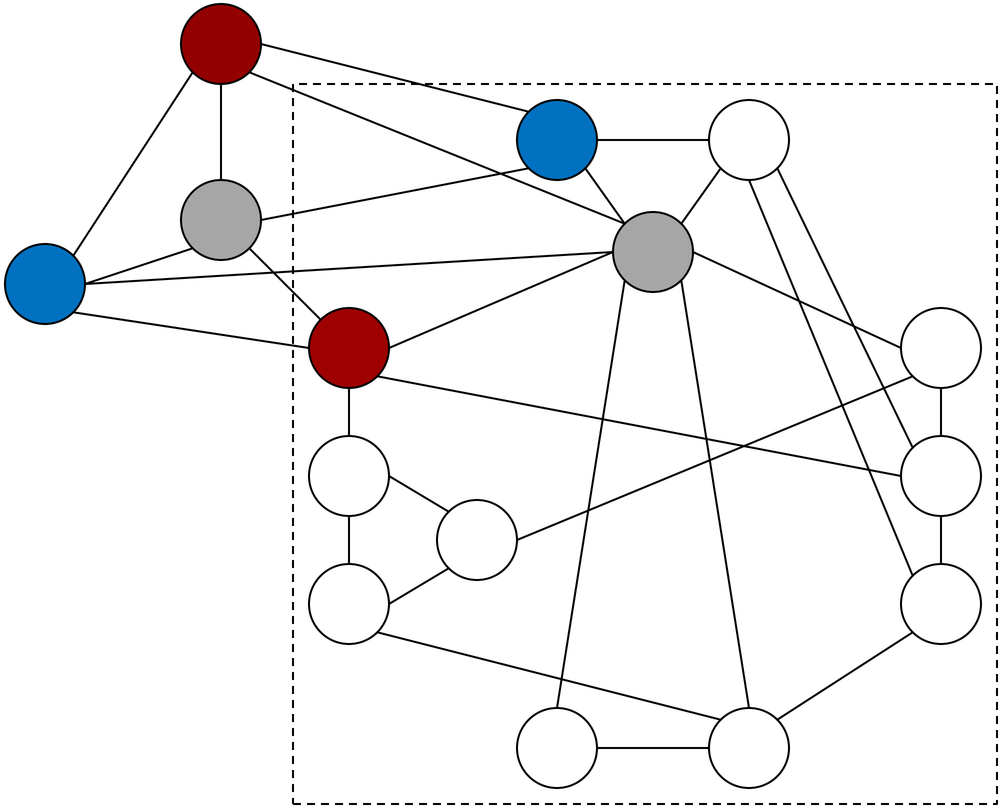
3-COLORABILITY ORACLES



YES

Given:
3-colorability decision oracle

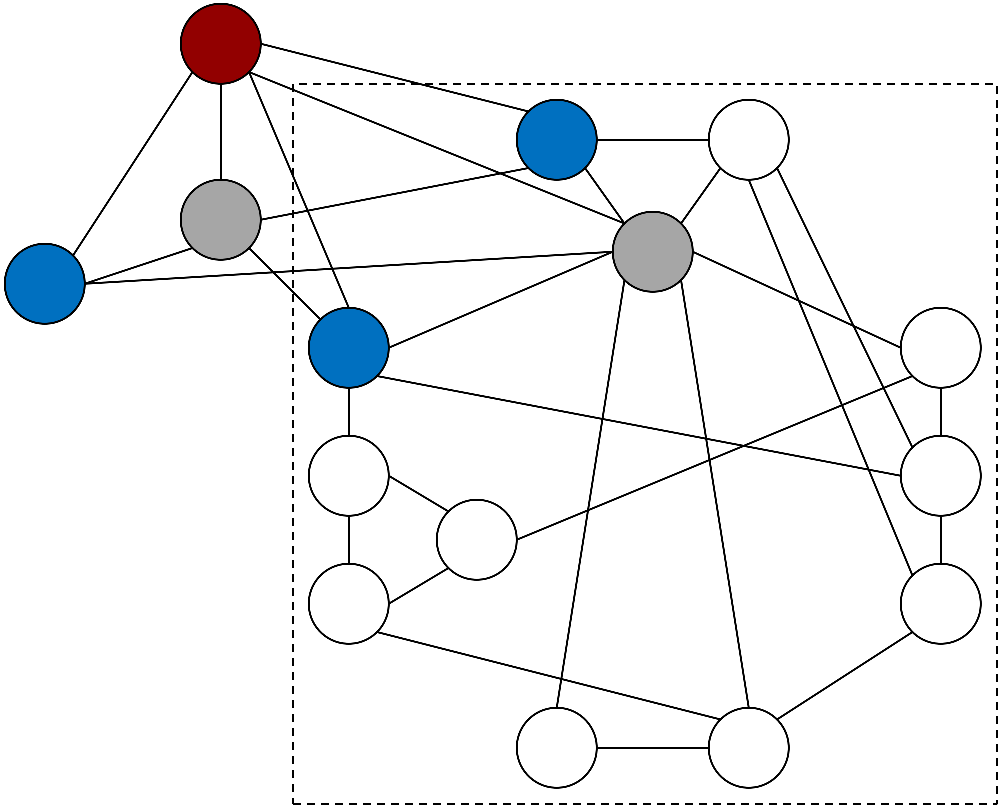
3-colorability oracles



Given:
3-colorability decision oracle

NO

3-colorability oracles



Given:
3-colorability decision oracle

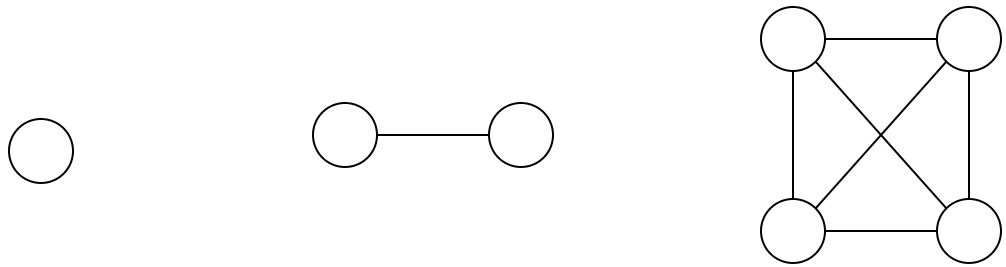
YES

3-colorability oracles

A 3-colorability search oracle can be simulated using a polynomial number of calls to a decision oracle!

CLIQUE

- A k-clique is a set of k nodes with all possible edges between them

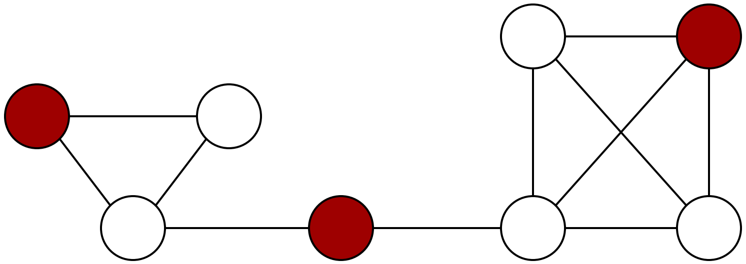


1-clique 2-clique 4-clique

- CLIQUE: Given a graph G and $k \in \mathbb{N}$, does G contain a k-clique?

Independent set

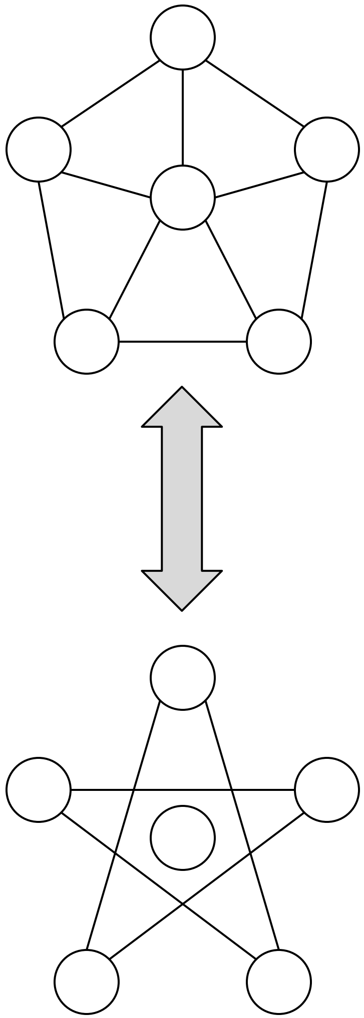
- A k-independent set is a set of k nodes with no edges between them



- INDEPENDENT-SET: Given a graph G and $k \in \mathbb{N}$, does G contain a k-independent set?

Clique vs. IS

- Let $G^* = (V, E^*)$ be the complement of $G = (V, E)$
  $$(u, v) \in E \Leftrightarrow (u, v) \notin E^*$$
- Vote: G has a k-clique for $k \geq 2$ iff:
  1. G^* has an IS of size $k - 2$
  2. G^* has an IS of size $k - 1$
  3. G^* has an IS of size k
  4. G^* has an IS of size $k + 1$



CLIQUE VS. IS

$\langle G, k \rangle$

YES/NO

Construct:
INDEP.-SET
decision oracle

$\langle G^*, k \rangle$

YES/NO

Given:
CLIQUE
decision oracle

Clique vs. IS

$\langle G, k \rangle$

YES/NO

Construct:
Clique
oracle

$\langle G^*, k \rangle$

YES/NO

Given:
Indep.-Set
oracle

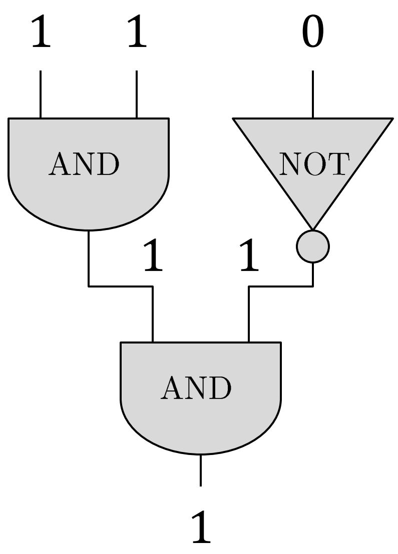
Clique vs. IS

- We can quickly reduce an instance of Clique to an instance of Independet-Set, and vice versa
- There is a fast method for one iff there is a fast method for the other

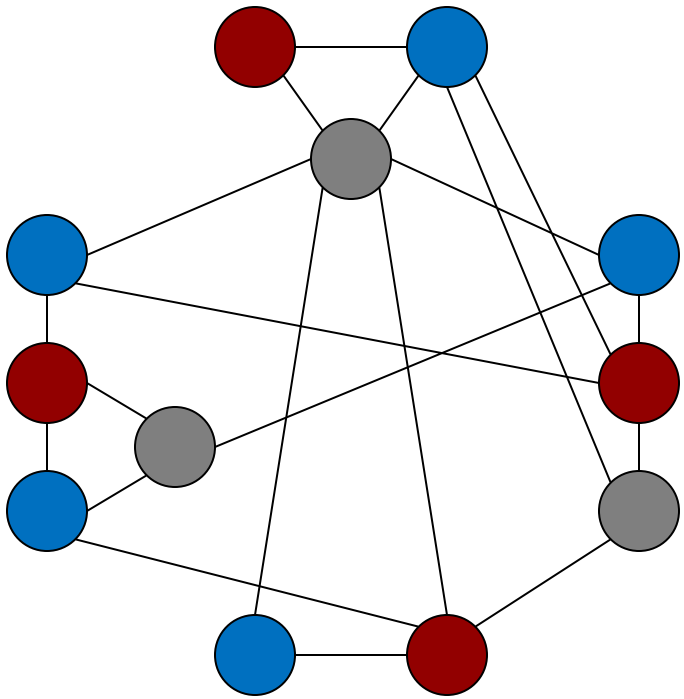
Clique and Independent-Set are cosmetically different but essentially the same!

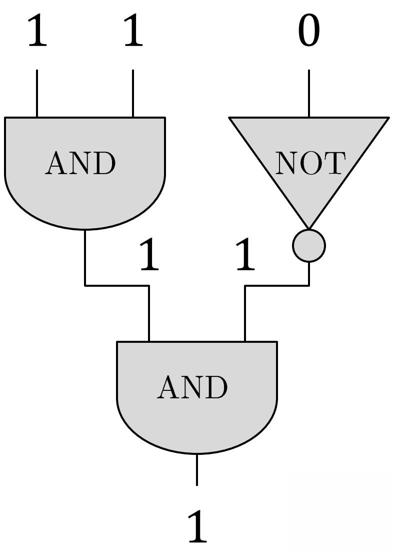
Circuit-Sat

- AND, OR, NOT gates wired together with no feedback allowed
- Circuit-Satisfiability: Given a circuit with n inputs and one output, is there a way to assign 0/1 values to the input wires so that the output value is 1 (true)?



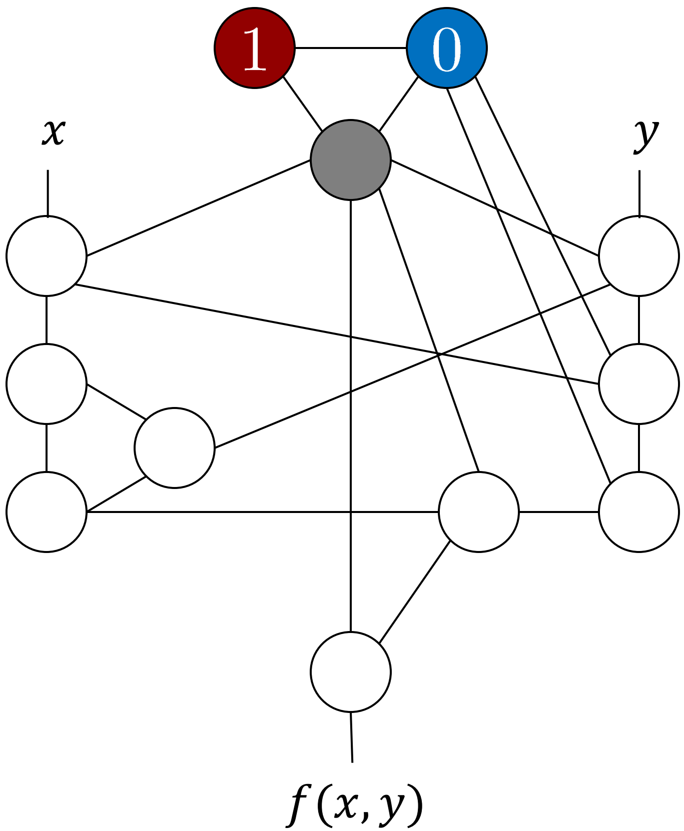
3-COLORABILITY VS. CIRCUIT-SAT





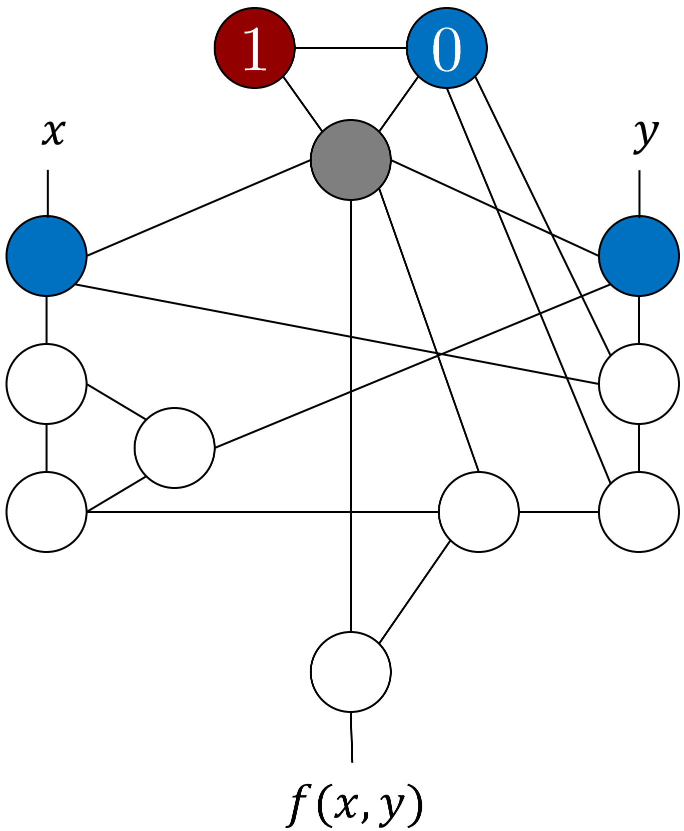
These two problems are fundamentally different!

3-colorability vs. circuit-Sat



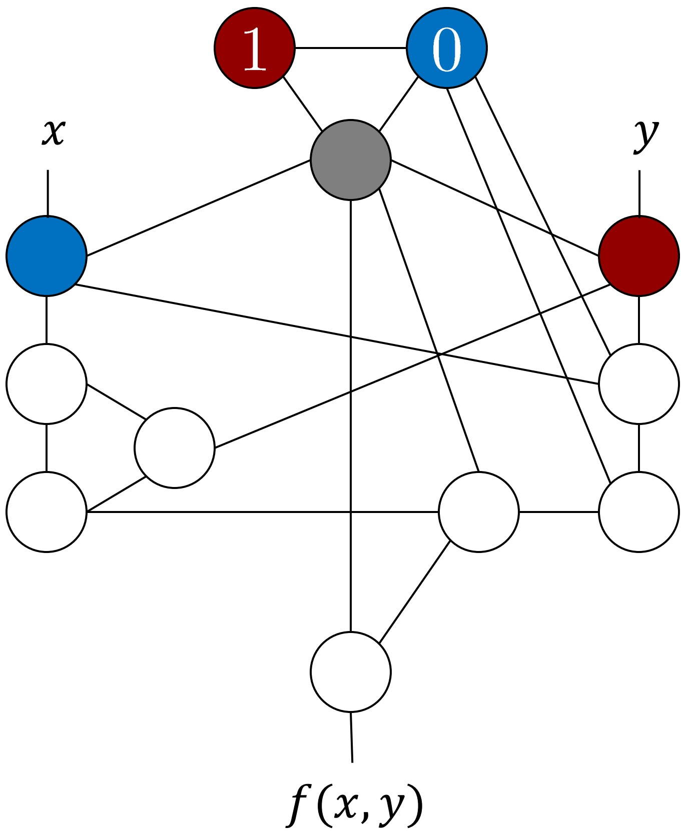
x	y	$f(x,y)$

3-colorability vs. circuit-Sat



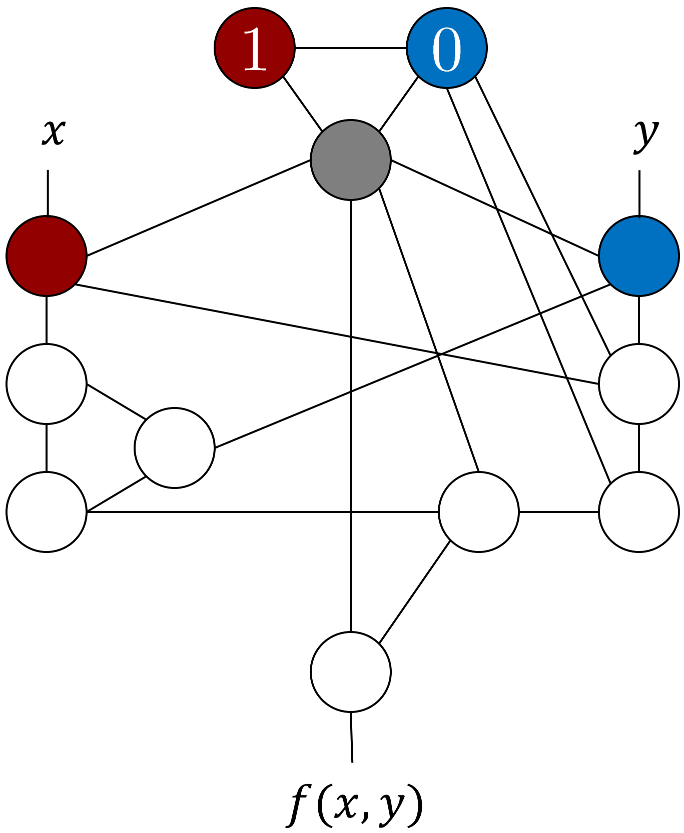
x	y	$f(x, y)$
0	0	0

3-colorability vs. circuit-Sat



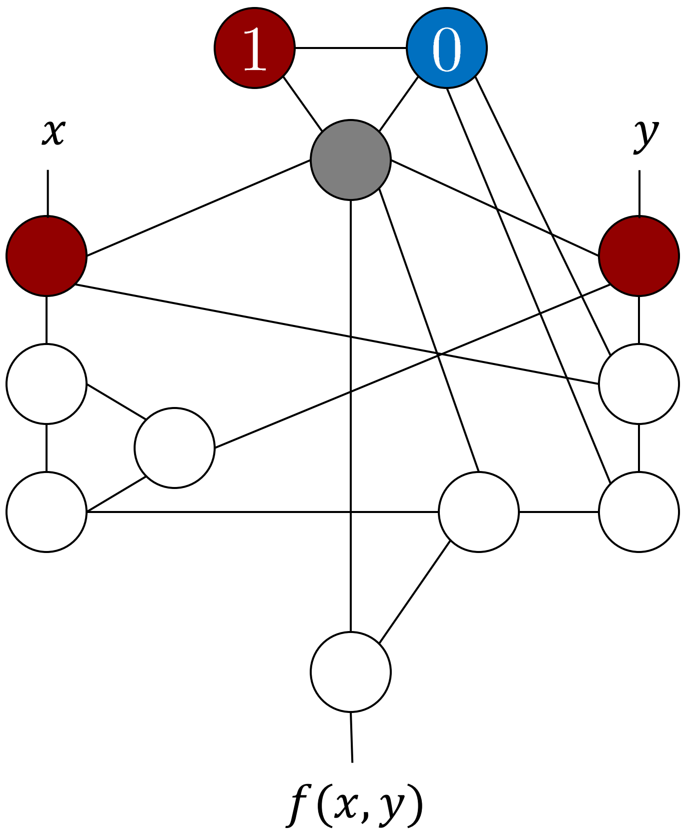
x	y	$f(x,y)$
0	0	0
0	1	1

3-colorability vs. circuit-Sat



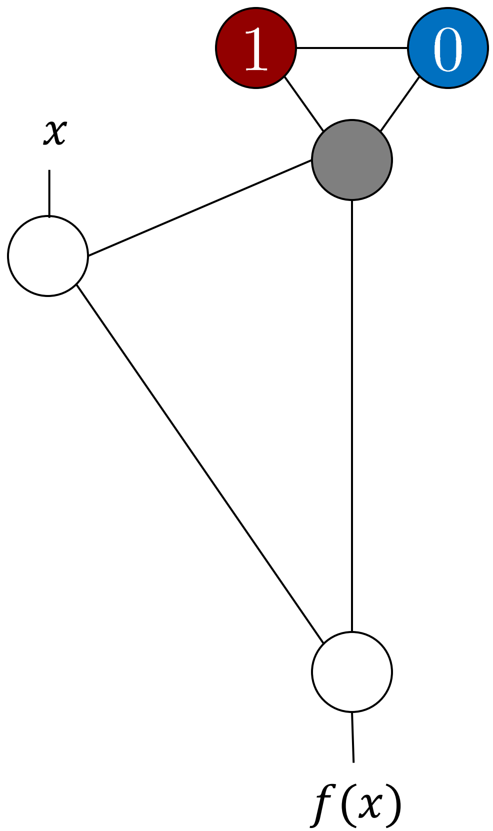
x	y	$f(x,y)$
0	0	0
0	1	1
1	0	1

3-colorability vs. circuit-Sat



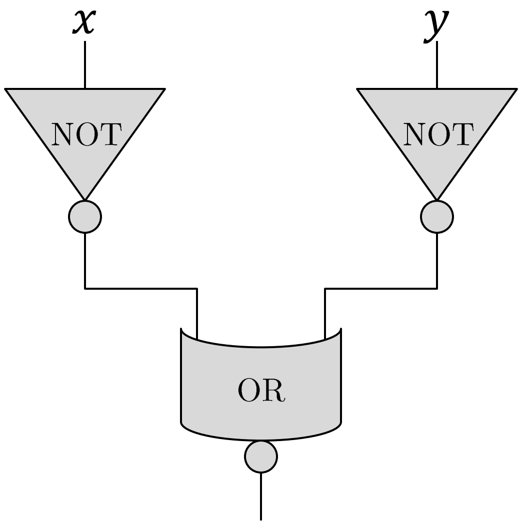
x	y	OR
0	0	0
0	1	1
1	0	1
1	1	1

3-colorability vs. circuit-Sat



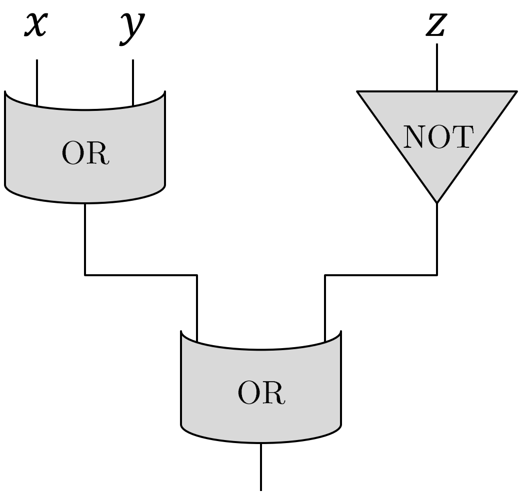
x	NOT
0	1
1	0

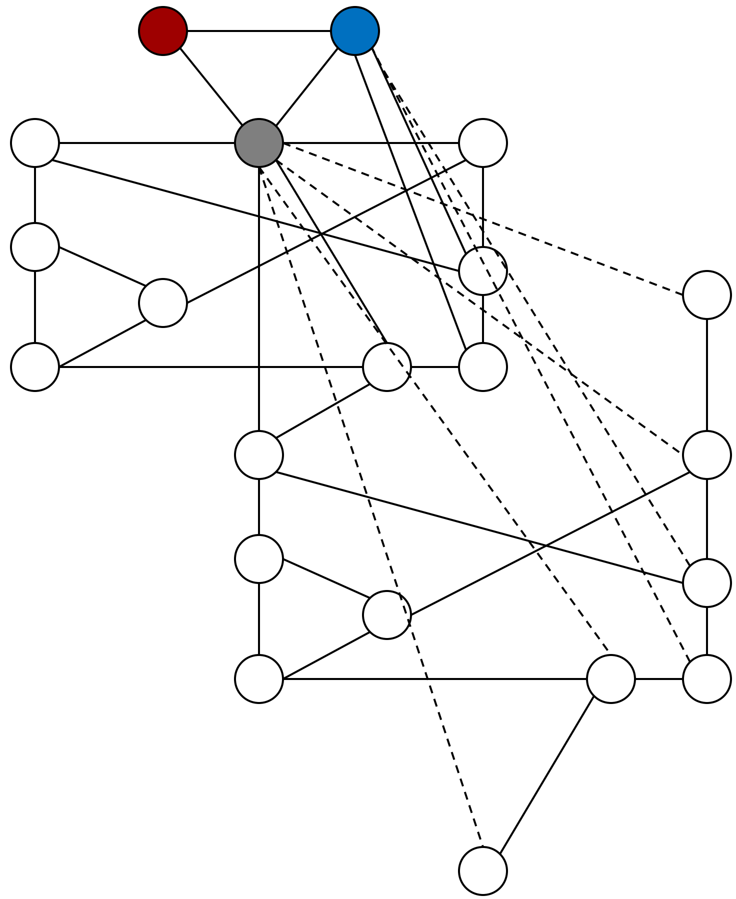
3-colorability vs. circuit-Sat



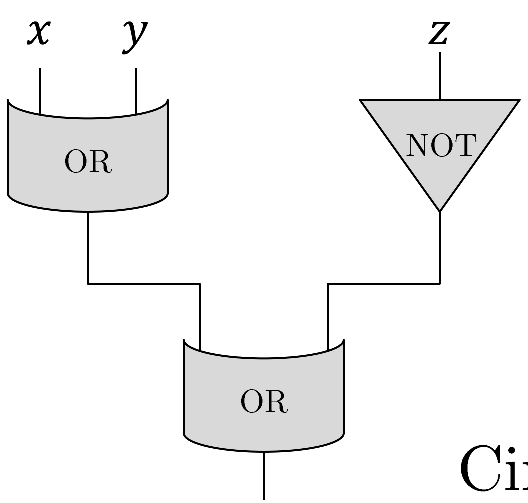
AND Gate from OR and NOT

3-colorability vs. circuit-Sat





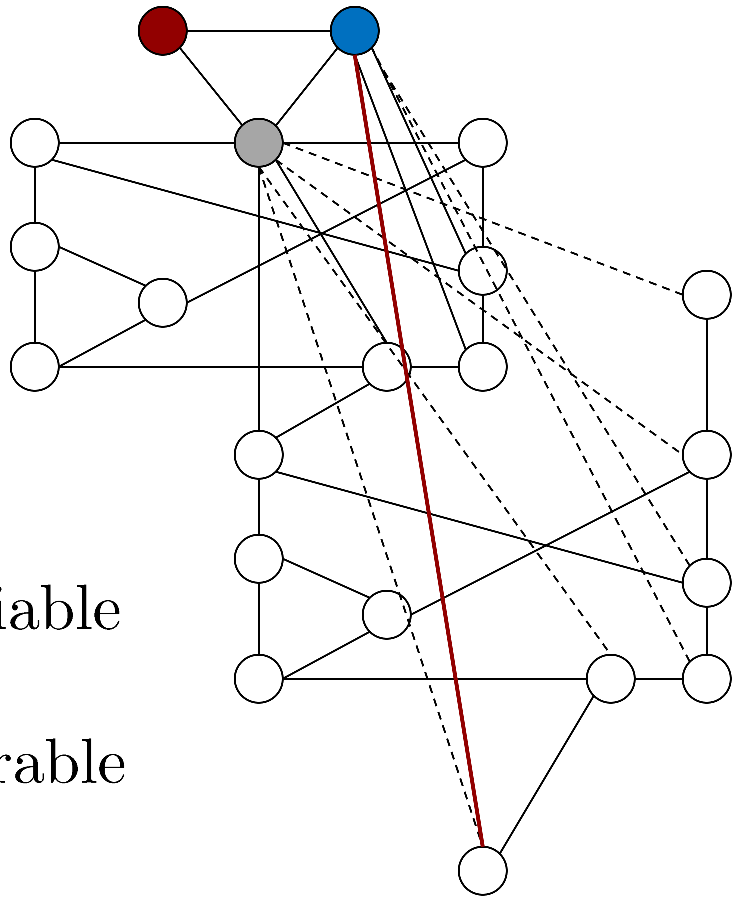
3-colorability vs. circuit-Sat



Circuit is satisfiable

⇕

Graph is 3-colorable



3-COLORABILITY VS. CIRCUIT-SAT

Circuit →

← YES/NO

Construct: CIRCUIT-SAT decision oracle

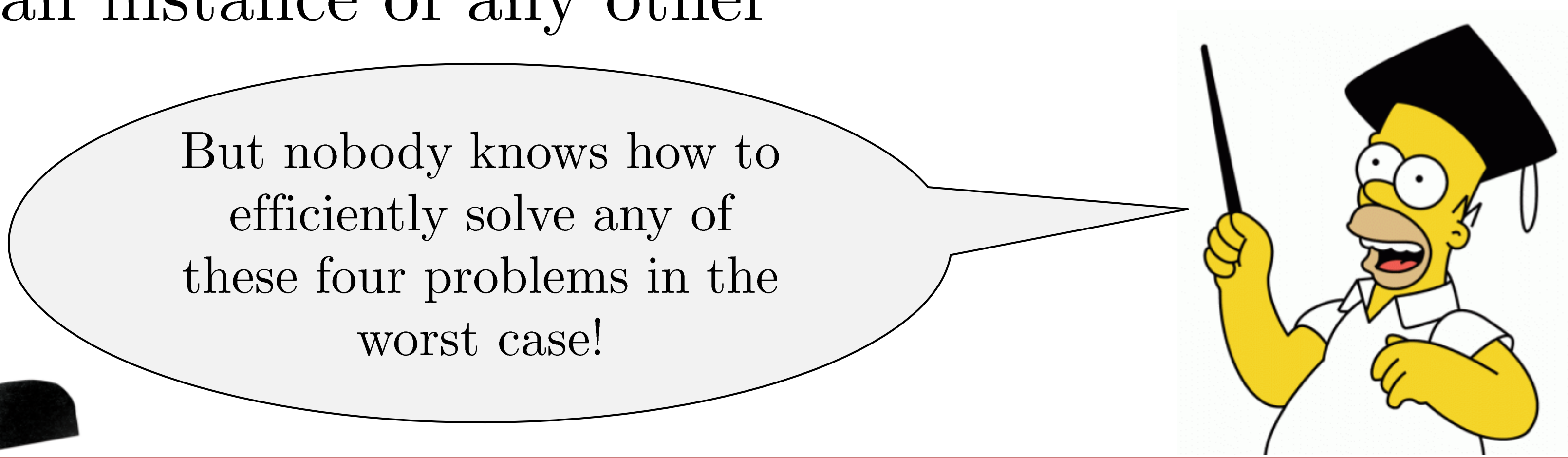
Graph with subgraphs corresponding to gates and edge between output and false →

← YES/NO

Given: 3-COLORAB. decision oracle

3-COLORABILITY VS. CIRCUIT-SAT

- There is a linear-time function that reduces instances of CIRCUIT-SAT to instances of 3-COLORABILITY
- **Fact:** There are efficient ways to reduce an instance of any of the four problems we discussed to an instance of any other



What we have learned

- Definitions:
  - k-COLORING
  - CLIQUE
  - INDEPENDENT-SET
  - CIRCUIT-SAT
- Principles:
  - Reduction between problems!