CMU 15-251
Cake cutting

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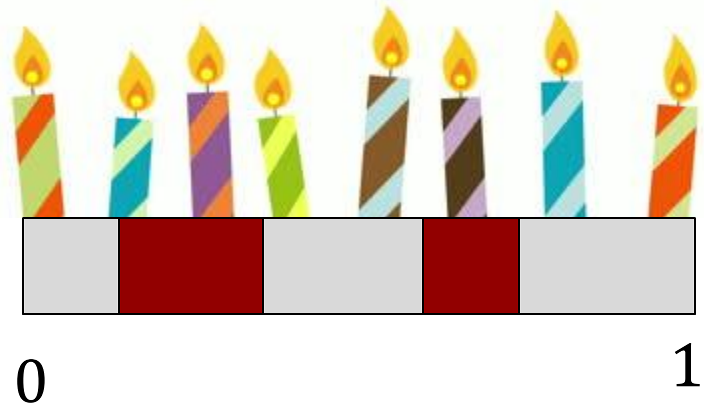
Cake cutting

- A cake must be divided between several children
- The cake is heterogeneous
- Each child has different value for same piece of cake
- How can we divide the cake fairly?
- What is "fairly"?
- A metaphor for land disputes, time using shared resources, etc.

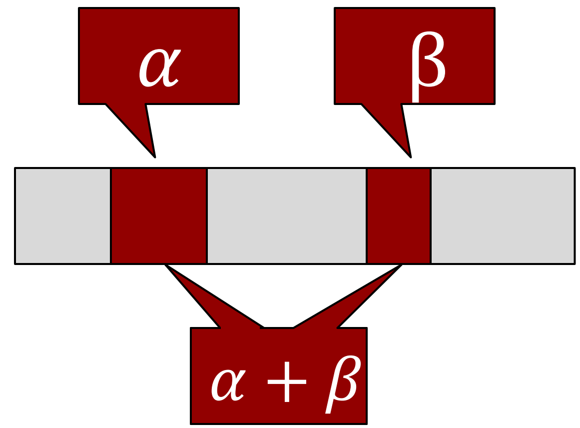
The problem

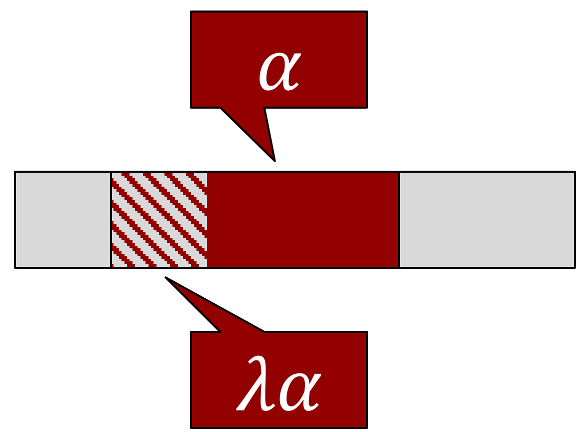
- Cake is interval [0,1]
- Set of players $N = \{1, \ldots, n\}$
- Piece of cake $X \subseteq [0,1]$: finite union of disjoint intervals



The problem

- Each player has valuation V_i over pieces of cake
- Additive: for $X \cap Y = \emptyset$, $V_i(X) + V_i(Y) = V_i(X \cup Y)$
- Normalized: For all $i \in N$, $V_i([0,1]) = 1$
- Divisible: $\forall \lambda \in [0,1]$ can cut $I' \subseteq I$ s.t. $V_i(I') = \lambda V_i(I)$





Fairness properties

- Our goal is to find an allocation $A_1, \ldots, A_n$
- Proportionality:

$$\forall i \in N, V_i(A_i) \geq \frac{1}{n}$$

- Envy-Freeness (EF):

$$\forall i, j \in N, V_i(A_i) \geq V_i(A_j)$$

- Vote: For $n = 2$ which is stronger?
  1. Proportionality
  2. EF
  3. They are equivalent
  4. They are incomparable

Fairness properties

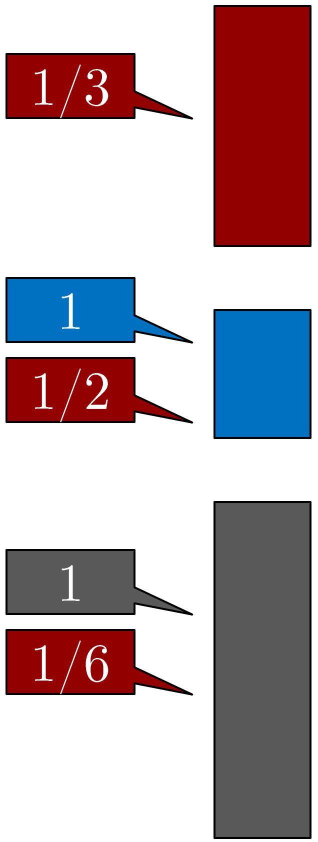
- Proportionality:

$$\forall i \in N, V_i(A_i) \geq \frac{1}{n}$$

- Envy-Freeness (EF):

$$\forall i, j \in N, V_i(A_i) \geq V_i(A_j)$$

- Vote: For $n \geq 3$ which is stronger?
    1. Proportionality
    2. EF
    3. They are equivalent
    4. They are incomparable

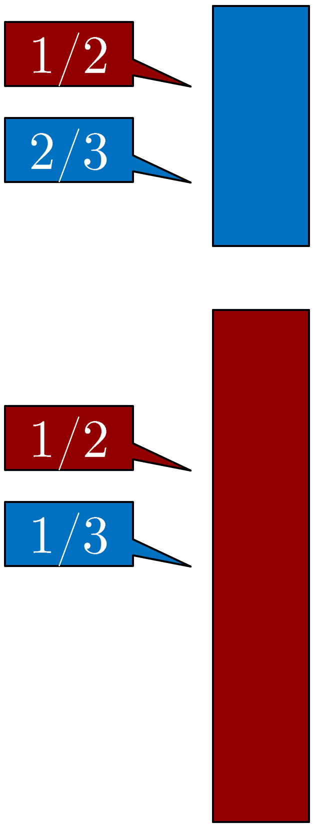


Cut-and-Choose

- Algorithm for $n = 2$ [Procaccia and Procaccia, circa 1985]
- Player 1 divides into two pieces X, Y s.t.

$$V_1(X) = 1/2\,, V_1(Y) = 1/2$$

- Player 2 chooses preferred piece
- This is EF (hence proportional)



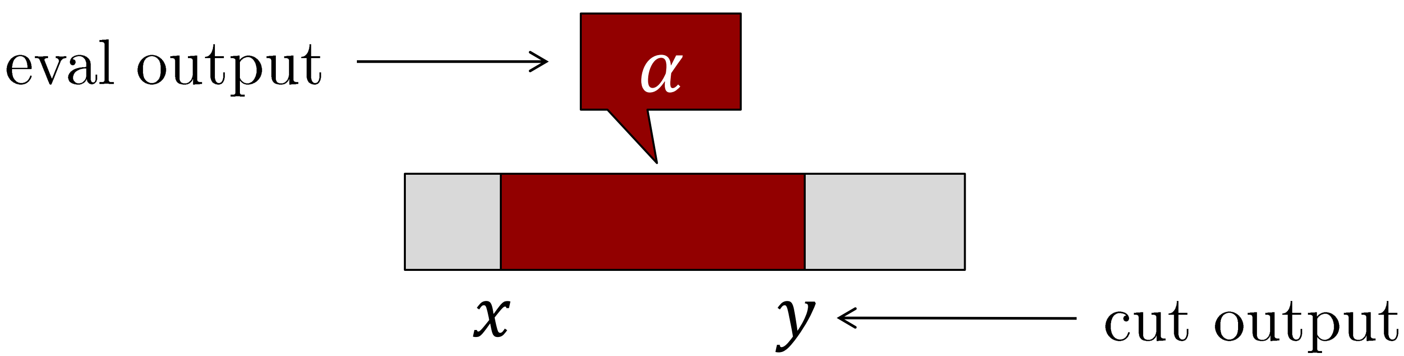
Time complexity

- Player 1 divides into two pieces X, Y s.t.
$$V_1(X) = 1/2, V_1(Y) = 1/2$$
- Player 2 chooses preferred piece

What is the running time of Cut-and-Choose? What is the input size?

The Robertson-Webb model

- Input size is n
- Two types of operations
  - $\mathtt{Eval}_i(x, y)$ returns $V_i([x, y])$
  - $\mathtt{Cut}_i(x, \alpha)$ returns y such that $V_i([x, y]) = \alpha$



The Robertson-Webb model

- Two types of operations
  - $\text{Eval}_i(x, y) = V_i([x, y])$
  - $\text{Cut}_i(x, \alpha) = y$ s.t. $V_i([x, y]) = \alpha$
- Vote: #operations needed to find an EF allocation when $n = 2$?
  1. 1
  2. 2
  3. 3
  4. 4

This concrete complexity model is a great idea!

Dubins-Spanier

- Referee continuously moves knife
- Repeat: when piece left of knife is worth $1/n$ to player, player shouts “stop” and gets piece
- That player is removed
- Last player gets remaining piece

Dubins-Spanier

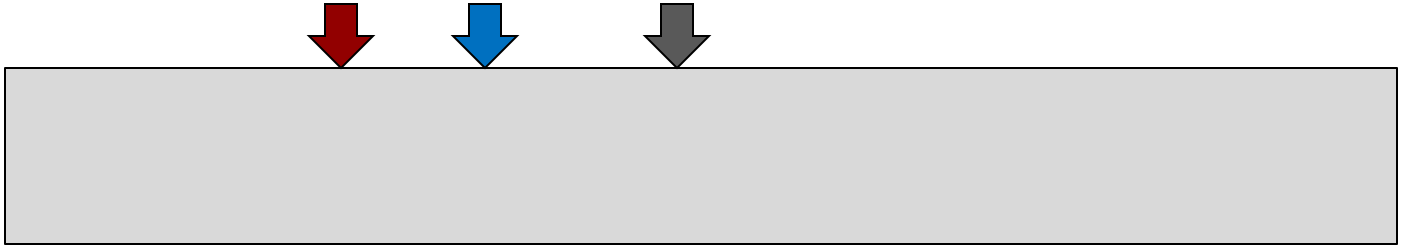
- **Claim**: The Dubins-Spanier protocol produces a proportional allocation
- **Proof:**
    - At stage 0, each of the n players values the whole cake at 1
    - At each stage, the allocated piece of cake is worth at most $1/n$ to the remaining players
    - Hence, if at stage k each of the remaining $n-k$ has value at least $1-\frac{k}{n}$ for the remaining cake, then at stage $k+1$ each of the remaining $n-(k+1)$ players has value at least $1-\frac{k+1}{n}$ for the remaining cake ■

Dubins-Spanier

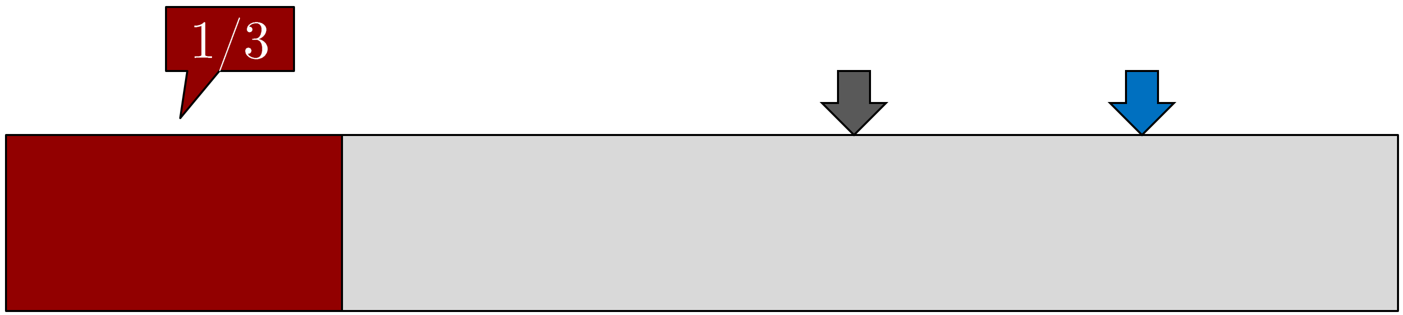
What is the complexity of Dubins-Spanier in the RW model?

- Moving knife is not really needed
- Repeat: each player makes a mark at his $1/n$ point, leftmost player gets piece up to its mark

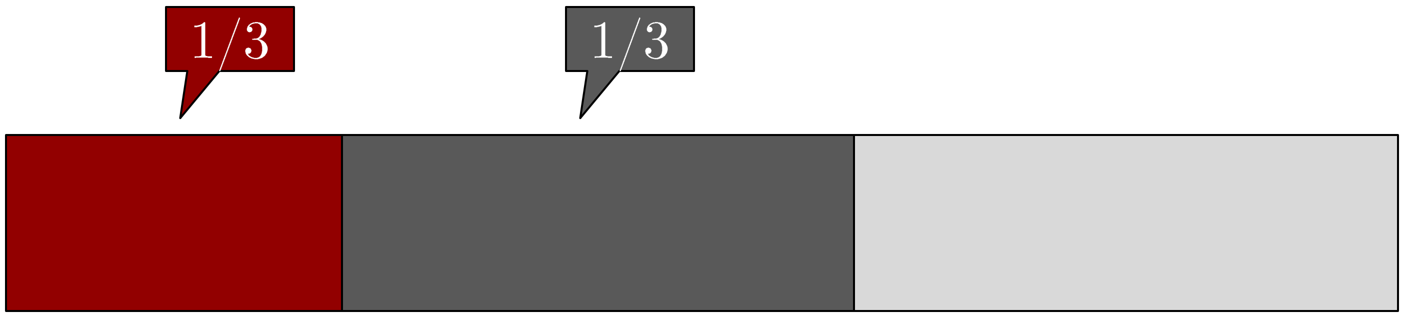
Dubins-Spanier



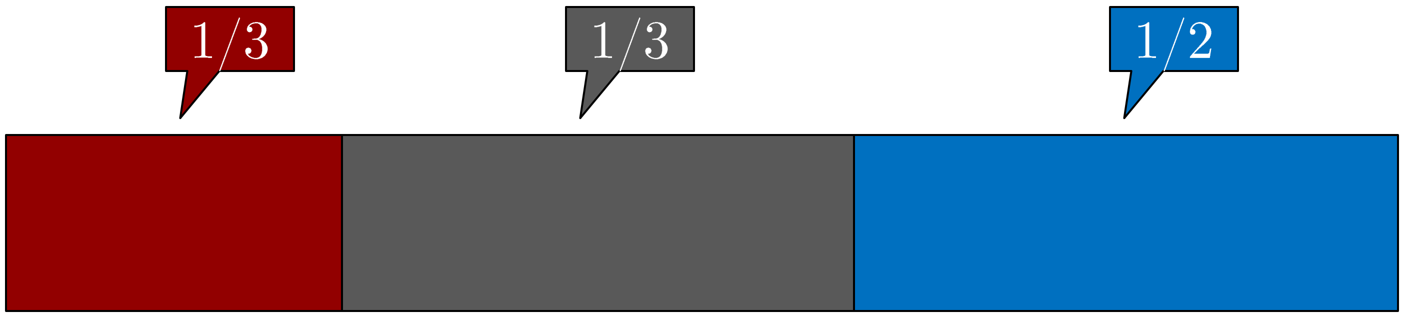
Dubins-Spanier



Dubins-Spanier



Dubins-Spanier



Dubins-Spanier

- Vote: So what is the complexity of Dubins-Spanier in the RW model?
    1. $\Theta(n)$
    2. $\Theta(n \log n)$
    3. $\Theta(n^2)$
    4. $\Theta(n^2 \log n)$

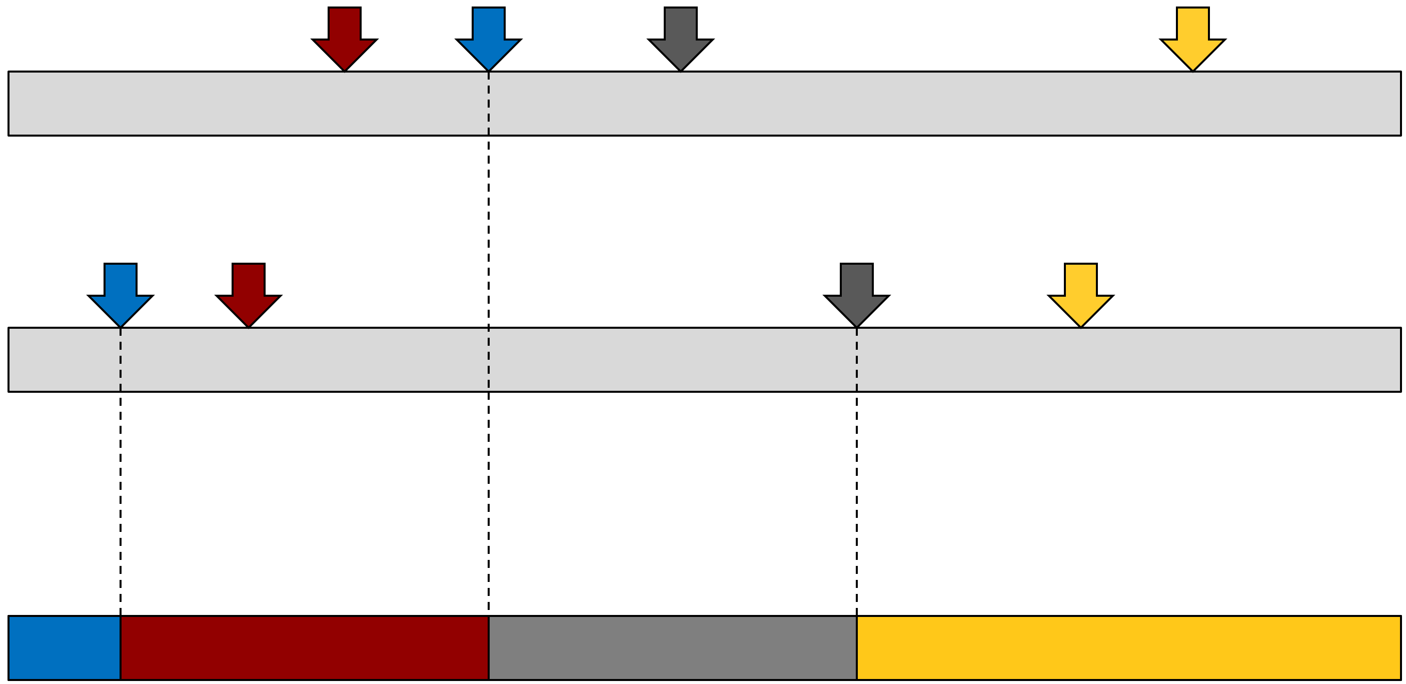
Even-Paz

- Given $[x, y]$, assume $n = 2^k$
- If $n = 1$, give $[x, y]$ to the single player
- Otherwise, each player i makes a mark z s.t.

$$V_i([x, z]) = \frac{1}{2} V_i([x, y])$$

- Let z^* be the $n/2$ mark from the left
- Recurse on $[x, z^*]$ with the left $n/2$ players, and on $[z^*, y]$ with the right $n/2$ players

Even-Paz



Even-Paz

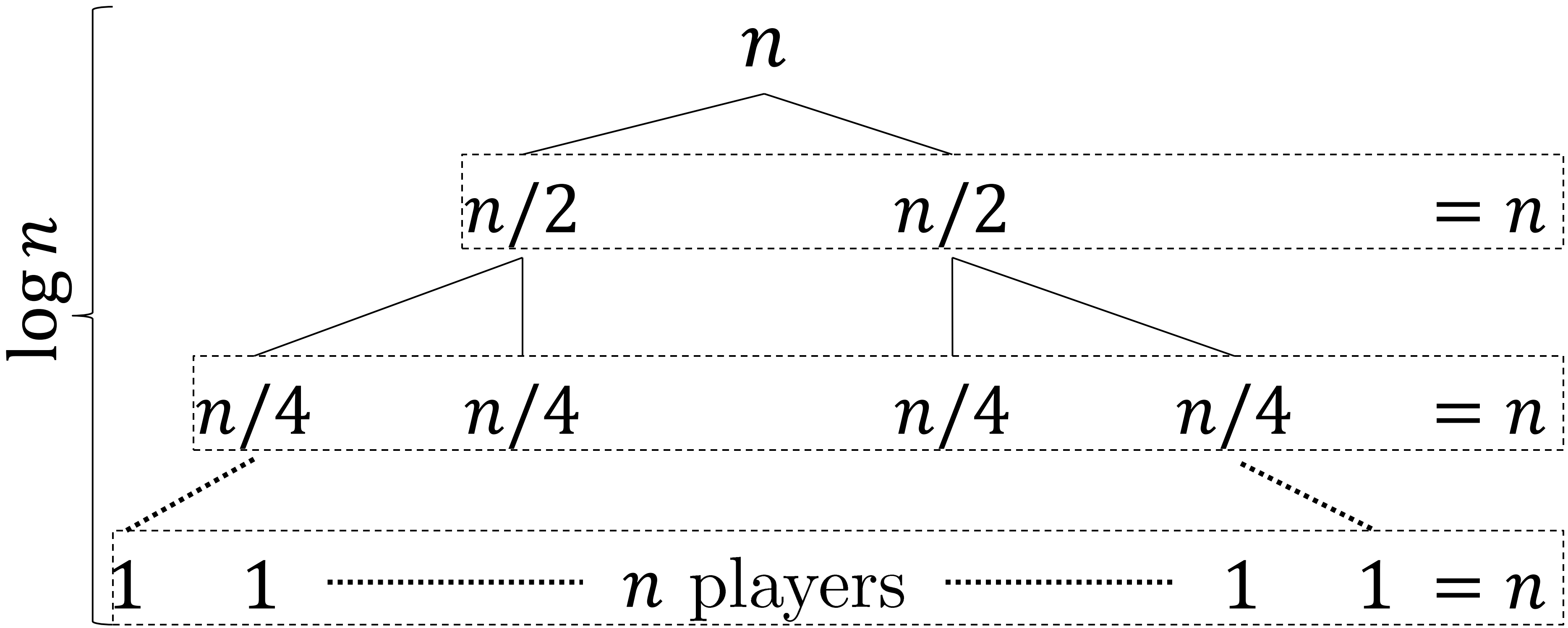
- **Claim**: The Even-Paz protocol produces a proportional allocation
- **Proof:**
  - At stage 0, each of the n players values the whole cake at 1
  - At each stage the players who share a piece of cake value it at least at $V_i([x,y])/2$
  - Hence, if at stage k each player has value at least $1/2^k$ for the piece he's sharing, then at stage $k+1$ each player has value at least $\frac{1}{2^{k+1}}$
  - The number of stages is $\log n$ ∎

Even-Paz

- Vote: What is the correct recursion formula for Even-Paz?

1. $T(1) = 0, T(n) = n + 2T\left(\frac{n}{2}\right)$
2. $T(1) = 0, T(n) = \frac{n}{2} + 2T\left(\frac{n}{2}\right)$
3. $T(1) = 0, T(n) = n + T\left(\frac{n}{2}\right)$
4. $T(1) = 0, T(n) = \frac{n}{2} + T\left(\frac{n}{2}\right)$

Even-Paz



Overall: $n \log n$

Complexity of proportionality

- **Theorem** [Edmonds and Pruhs, 2006]: Any proportional protocol needs $\Omega(n \log n)$ operations in the RW model
- The Even-Paz protocol is provably optimal!

What we have learned

- Definitions:
  - Proportionality / envy-freeness
  - The Robertson-Webb model
  - The Dubins-Spanier protocol
  - The Even-Paz protocol
- Principles:
  - Concrete complexity models for reasoning about time complexity