Great Theoretical Ideas In Computer Science

Victor Adamchik CS 15-251

Lecture 8 Carnegie Mellon University

Graphs - I

Plan

- Graph Representations
- Counting Trees
- Cayley's Formula
- Prüfer Sequence
- Minimum Spanning Trees
- Planar Graphs
- Euler's Polyhedra Theorem

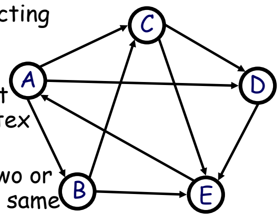
Definition

A graph G is a pair (V,E) where
V is a set of vertices (or nodes)
E is a set of edges connecting the vertices

A self-loop is an edge that connects to the same vertex twice

A multi-edge is a set of two or more edges that have the same two vertices

A graph is simple if it has no multi-edges or self-loops.



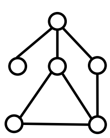
More terms

Directed: an edge is an ordered pair of vertices
Undirected: edge is unordered pair of vertices
Weighted: (a cost associated with an edge)
Path (is a sequence of no-repeated vertices)
Cycle (the start and end vertices are the same)
Acyclic
Connected or Disconnected
The degree of a vertex (in an undirected graph is the number of edges associated with it.)

The handshaking theorem

Let G=(V,E) be an undirected graph with V vertices and E edges. Then

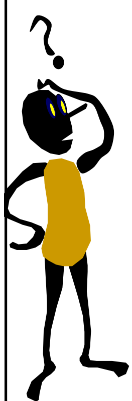
$$2E = \sum_{x \in V} \deg(x)$$



In a directed graph:

$$E = \sum_{x \in V} \mathrm{indeg}(x) = \sum_{x \in V} \mathrm{outdeg}(x)$$

Exercise



Given a graph with 7 vertices; 3 of them of degree two and 4 of degree one. Is this graph is connected?

$$2E = \sum_{x \in V} \deg(x)$$

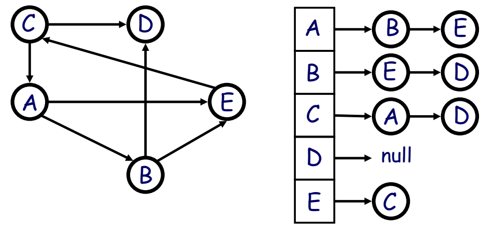
No, the graph has only 5 edges.

Representing Graphs

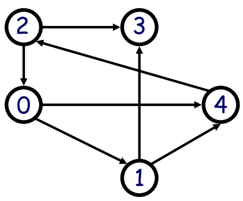
Adjacency List
or
Adjacency Matrix

Vertex X is *adjacent* to vertex Y if and only if there is an edge (X, Y) between them.

Adjacency List Representation



Adjacency Matrix Representation



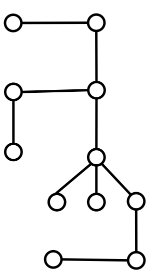
$$\begin{pmatrix} 0&1&0&0&1 \\ 0&0&0&1&1 \\ 1&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&1&0&0 \end{pmatrix}$$

Representing Graphs

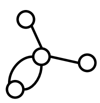
Adjacency List Representation is used for representation of the sparse graphs.

Adjacency Matrix Representation is used for representation of the dense graphs.

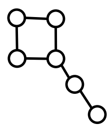
Trees



A tree is a connected simple graph without cycles.



Not Tree



Not Tree

<u>Theorem</u>: Let *G* be a graph with V nodes and E edges

The following are equivalent (TFAE) :

1. *G* is a tree (connected, acyclic)
2. Every two nodes of *G* are joined by a unique path
3. *G* is connected and V = E + 1
4. *G* is acyclic and V = E + 1
5. *G* is acyclic and if any two non-adjacent nodes are joined by an edge, the resulting graph has exactly one cycle

To prove this, it suffices to show
$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$

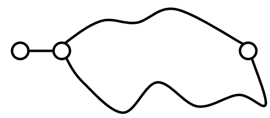
We'll just show
$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$
and leave the rest to the reader

$1 \Rightarrow 2$

1. *G* is a tree (connected, acyclic)
2. Every two nodes of *G* are joined by a unique path

Proof: (by contradiction)

Assume *G* is a tree that has two nodes connected by two different paths:



Then there exists a cycle!

$2 \Rightarrow 3$

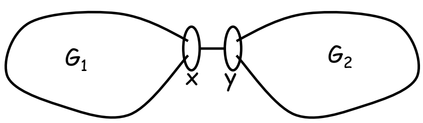
2. Every two nodes of *G* are joined by a unique path

3. *G* is connected and V = E + 1

Proof: (by strong induction)

Assume true for every graph with < V vertices

Let *G* have V nodes and let x and y be adjacent



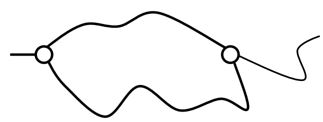
Then $V = V_1 + V_2 = E_1 + E_2 + 2 = E + 1$

$3 \Rightarrow 4$

3. *G* is connected and V = E + 1
4. *G* is acyclic and V = E + 1

Proof: by contradiction

Assume, *G* has a cycle with k vertices.



Start adding nodes and edges until you cover the whole graph. Number of edges in the graph will be at least V, since the cycle has k vertices and k edges.

Corollary: Every nontrivial tree has at least two vertices of degree 1.

Proof (by contradiction):

Assume all but one of the vertices in the tree have degree at least 2. Can we?

In any graph, sum of the degrees = 2E

Under our assumption $2E = \sum deg_i \geq 2(V-1)+1$

Then the total number of edges in the tree is at least $E \geq (2V-1)/2 = V - 1/2 > V - 1$

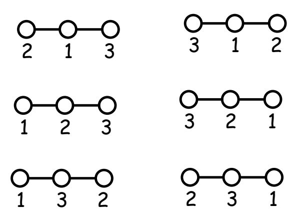
Contradiction, since in a tree E = V - 1

Cayley's Formula

Using the above property, we can now begin to discuss Cayley's formula that tells us how many different trees we can construct on n vertices.

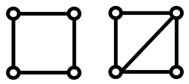
How many labeled trees are there with three nodes?

Two labeled trees with the same set of labels are isomorphic iff they have the same adjacency matrix.

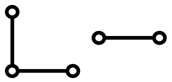


Graph Isomorphism

Definition. Two simple graphs G and H are isomorphic $G \cong H$ if there is a vertex bijection $V_H \to V_G$ that preserves adjacency and non-adjacency structures.



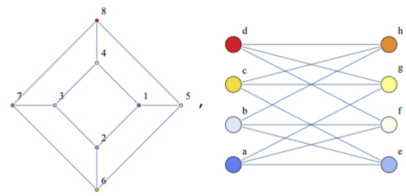
Does not preserve adjacency



It's not bijective

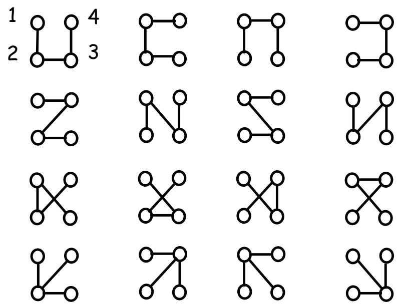
Graph Isomorphism

The graph isomorphism problem has no known polynomial time algorithm which works for an arbitrary graph.

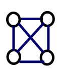


1->a, 2->e, 3->b, 4->f, 5->g, 6->c, 7->h, 8->d

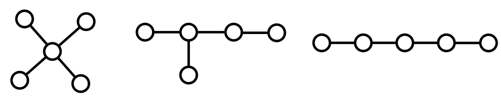
How many labeled trees are there with four nodes?



These are called spanning trees, for a complete graph of 4 vertices.



How many labeled trees are there with five nodes?



5 labelings

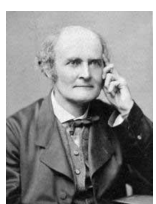
5 x 4 x 3 labelings

5!/2 labelings

125 labeled trees

Cayley's Formula (1889)

The number of labeled trees on n nodes is n^{n-2}



Arthur Cayley (1821-1895)

Put another way, it counts the number of spanning trees of a complete graph

Prüfer Encoding (1918)

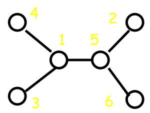
We are going to find a <u>bijection</u> between the set of sequences and the set of labeled trees.

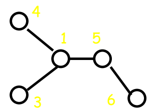
A Prüfer sequence is a sequence of n−2 numbers, each being one of the numbers 1 through n. We should initially note that indeed there are n^{n-2} Prüfer sequences for any given n.

bijection: T(n) -> P(n-2)

Encoding a tree into a Prüfer sequence

Take a tree and label vertices from 1 to n in any manner.



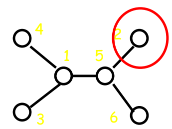


Take the vertex with the smallest label whose degree is equal to 1, delete it from the tree and write down the value of its only *neighbor*.

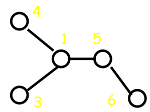
Repeat this process until only two vertices remain.

So now we have a sequence of n – 2 elements encoded from our tree.

Encoding a tree into a Prüfer sequence

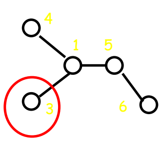


Sequence:

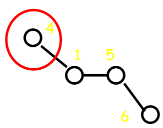


Sequence: 5

Encoding a tree into a Prüfer sequence

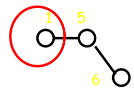


Sequence: 5, 1

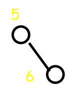


Sequence: 5,1,1

Encoding a tree into a Prüfer sequence

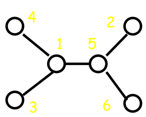


Sequence: 5, 1, 1, 5



Sequence: 5, 1, 1, 5

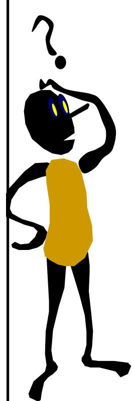
Encoding a tree into a Prüfer sequence



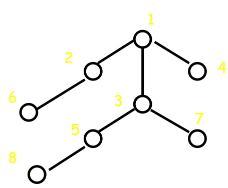
P = 5, 1, 1, 5

Notice that all of the vertices of degree 1 do not occur in P. No leaves in P.

Every vertex in P has degree equal to 1 + r, where r is the number of times that vertex appears in our sequence P.



Exercise: write the Prüfer sequence



Sequence: 1, 2, 1, 3, 3, 5

Reconstructing a tree

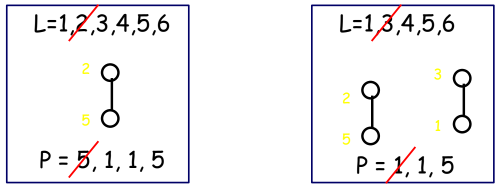
Given P = $\{a_1,\ldots,a_{n-2}\}$ and the list L = {1,…, n}

Let k be the smallest number in L that is not in P.
Let a_j be the fist number in the Prüfer sequence P.
Connect k and a_j with an edge.
Remove k from L and a_j from P.

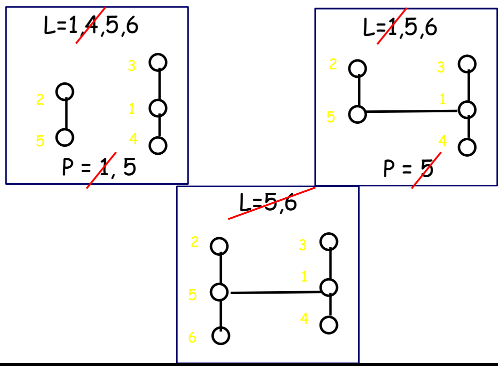
Repeat this process until all elements of P have been exhausting (n-2 times)

Connect the last two vertices in L with an edge.

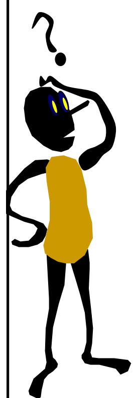
Reconstructing a tree



Reconstructing a tree



Exercise

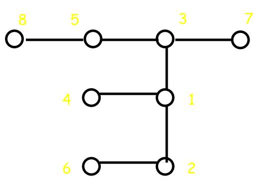


Given P = {1, 2, 1, 3, 3, 5}.
Reconstruct a tree.

Exercise

L = {1,2,3,4,5,6,7,8}

P = {1,2,1,3,3,5}



Bijection between Prüfer Sequences and Labeled Trees

Let T be a set of labeled tree of n vertices
Let P be a set of Prüfer sequences of length n-2

A map f: T-> P is a bijection.

A map f: T-> P is injective.

We need to show that two different trees T_1 ,T_2 generate different Prüfer sequences.
By Induction on the number of vertices.

Base case: n = 2, two vertices joined by an edge.

Assume it's true for n, prove it for n+1.

Take the lowest-labeled leaf in T_1 and in T_2.

Case 1: Those two leaves are different

Case 2: Same, but neighbors not

Case 3: Leaves and neighbors are the same

A map f: T-> P is surjective.

We need to show that any sequence $P=\{a_1,\dots,a_{n-2}\}$ generates at least one tree on $L=\{1, \dots, n\}$
By Induction on the number of vertices.

Base case: n = 2, P = {}.

Assume it's true for n, prove it for n+1.

Take the lowest $v_k \in L$ s.t. $v_k \notin P$

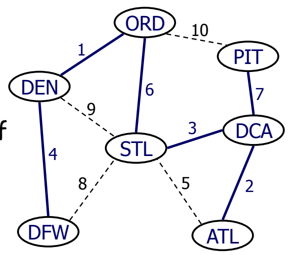
Consider $P' = P \backslash a_1$ and $L' = L \backslash v_k$. By IH there is T'.

Form T from T' by adding v_k joined with a_1.

Since a_1 is internal, T is a tree.

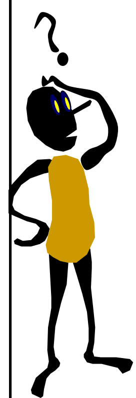
The Minimum Spanning Tree

Minimum spanning tree (MST) is a spanning tree of a weighted graph with minimum total edge weight



The weight of a spanning tree is the sum of the weights on all the edges which comprise the spanning tree.

The MST



Fred Hacker's algorithm:

Find ALL spanning trees and then pick one with the minimum cost.

What's wrong with this idea?

The number of spanning trees in K_n is n^{n-2}

The Minimum Spanning Tree

Joseph Kruskal (1929-2010)

Boruvka's Algorithm (1926)
Kruskal's Algorithm (1956)
Prim's Algorithm (1957)

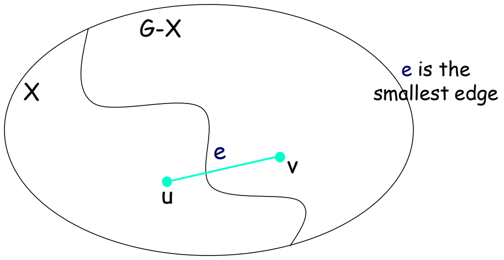
Robert Prim (1921-)

Property of the MST

Lemma: Let X be any subset of the vertices of G, and let edge e be the smallest edge connecting X to G-X. Then e is part of the minimum spanning tree.

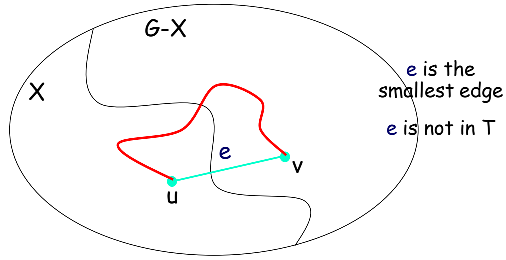
Property of the MST - proof

Let T be the MST && e not in T



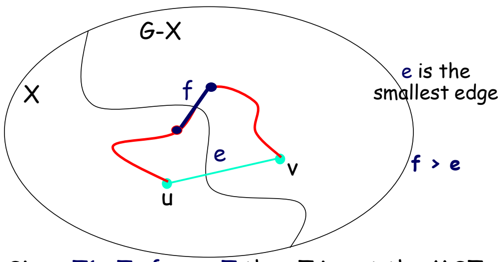
Property of the MST - proof

There exists a unique path in T from u to v.



Property of the MST - proof

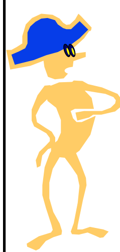
Let T be the MST && e not in T

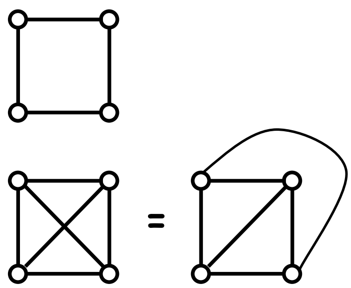


Since $T1 = T - f + e < T$ thus T is not the MST

Planar Graphs

A graph is planar if it can be drawn in the plane without crossing edges



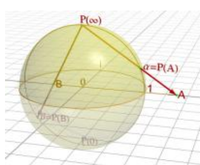


Planar Graphs

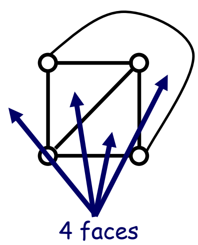
A graph is planar if it can be drawn in the plane without crossing edges

A graph is planar if and only if it can be embedded in a <u>sphere</u>. This is useful because often a sphere is more convenient to work with.

A sphere can be 1-1 mapped (except 1 point) to the plane and vice-versa. E.g. the stereographic projection:



Faces



A planar graph when drawn in the plane, splits the plane into disjoint faces.

Euler's Formula

If G is a connected planar graph with V vertices, E edges and F faces, then

$V - E + F = 2$

Generalized for any polyhedron,
For a cube:
v=8
e=12
f=6

Proof of Euler's Formula

For connected arbitrary planar graphs $V - E + F = 2$

The proof is by induction on edges.

Start with a single edge and 2 vertices:
V=2, E=1, F=1. Check.

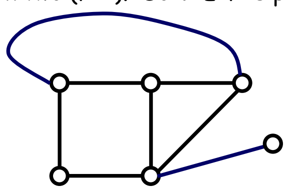
Add the edges in an order so that what we've added so far is connected.

There are two cases to consider.

---

(1) The edge connects two vertices already there.
(2) The edge connects the current graph to a new vertex

In case (1) we add a new edge (E++) and we split one face in two (F++). So V-E+F is preserved.

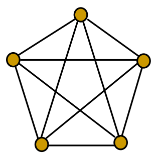


In case (2) we add a new vertex (V++) and a new edge (E++). So again V-E+F is preserved.

Theorem: In any connected planar graph with at least 3 vertices:

$E \leq 3V - 6$ $\qquad$ $E = O(V)$

By means of this theorem we can prove, for example, that a complete graph K_5 is not planar



K_5 has 5 vertices and 10 edges, thus

$E = 10 \leq 3x5 - 6 = 9$

which is clearly false

Theorem: In any connected planar graph with at least 3 vertices:

$E \leq 3V - 6$

Proof.

1. If the graph has no cycles,

$E = V-1 \leq V \leq V + (2V-6) = 3V - 6,$

since $V \geq 3$, and therefore $2V-6 \geq 0$,

Theorem: In any connected planar graph with at least 3 vertices:

$E \leq 3V - 6$

Proof (cont.)

2. If the graph has a cycle. We will count the number of pairs (edge, face).

Each face is bounded by at least 3 edges:

$\sum(\text{edge, face}) \geq 3F$

Each edge is associated with at most 2 faces:

$\sum(\text{edge, face}) \leq 2E$

It follows, $\quad 3F \leq 2E$

Theorem: In any connected planar graph with at least 3 vertices:

$$E \leq 3V - 6$$

Proof (cont.) We found, $3F \leq 2E$

By Euler's theorem :

$$2 = V - E + F$$

$$6 = 3V - 3E + 3F \leq 3V - 3E + 2E = 3V - E$$

QED

Planar Graphs

Theorem: In any connected planar graph with V vertices, E edges and F faces, then

$$V - E + F = 2$$

Theorem: In any connected planar graph with at least 3 vertices:

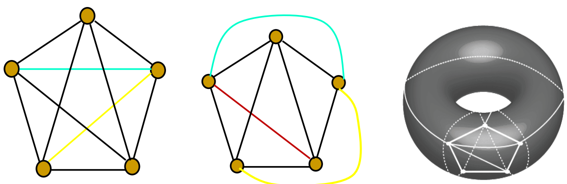
$$E \leq 3V - 6$$

Lemma: In any connected planar graph with at least 3 vertices:

$$3F \leq 2E$$

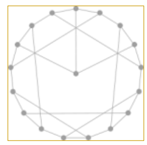
K_5 can be embedded on the torus

Embedding a graph onto a surface means drawing the graph on the surface such that no edges cross.

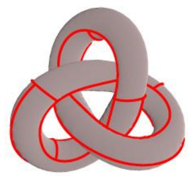


Always there is a surface so any graph can be embedded to.

More embeddings



Blanuša graph on a trefoil knot



Graph Isomorphism
Cayley's Formula
Prüfer Encoding
Minimum Spanning Trees
Planar Graphs
Euler's Polyhedra Theorem

Here's What You Need to Know…