15-251 : Great Theoretical Ideas In Computer Science

Fall 2013

Assignment 4 (The Kitten Homework)

Due: Thursday, Sep. 26, 2013 11:59 PM

Name: _____

Andrew ID: _____

Question:	1	2	3	4	5	6	7	8	Total
Points:	30	10	10	20	15	15	10	0	110
Score:									

1. Mrs. Ford's Kittens

(15) (a) My cat, Mrs. Ford, has just had kittens! A *lot* of kittens, say at least n male kittens and at least n female kittens. To wash her kittens, Mrs. Ford decides to follow this procedure: first, she washes one male and one female kitten. Then, at every time, she chooses an already-washed kitten (every already-washed kitten is equally likely to be chosen), and if the chosen kitten is male, washes another male kitten, or if the chosen kitten is female, washes another female kitten. Show that after first n kittens have been washed, where $n \geq 2$, the probability that exactly k female kittens have been washed is $\frac{1}{n-1}$ for all $1 \leq k \leq n-1$.

Solution:

(15) (b) Mrs. Ford, having washed all of her kittens, decides to sort n of them by size. The kittens have been running around and tripping all over themselves (as kittens do), so they are initially in a completely random order. Mrs. Ford sorts her kittens as follows: first, she picks a random kitten (each kitten has equal probability to be "the chosen one"). Then, she compares all other kittens' size with that of the chosen one, and puts all kittens smaller than the chosen one to its left, and all the rest to its right. Then, she follows the same method recursively to sort kittens on the left and on the right side separately.

My two favorite kittens, Pounce and Boots, are among the n being sorted. I really want to take a picture of Pounce and Boots together, but I can only do this if Mrs. Ford directly compares their sizes, making them stand next to each other for a moment. I know that there are i kittens smaller than Pounce and j kittens smaller than Boots (where $i < j$, since Boots is larger than Pounce), what is the probability that I will have an opportunity to take the picture?

Solution:

2. Kittens and Threads

(10) (a) Mrs. Ford's kittens are very fascinated by threads. Mrs. Ford has given them n threads to play with. The kittens start playing the following game: At each step, if k is the number of threads remaining (initially $k = n$), then a kitten chooses two of the $2k$ free ends randomly (each pair of free ends is chosen with equal probability among all pairs), and ties them together. Two things can happen:

1. This operation merges two distinct threads into one.
2. The kitten ends up tying two ends of the same thread together, into a loop. Kittens like loops! If this happens, the kitten takes the loop, and starts playing with it.

In any case, the number of threads remaining to play with decreases by 1. The game ends when there are no more threads to play with. Find the expected number of loops created at the end of this game.

Solution:

3. Superstitions

(10) (a) My neighbor is superstitious, but she's a bit strange. She believes that if k black cats cross her path in a row, without a cat of any other color crossing her path in between, she will be doomed! If every hour, one of Mrs. Ford's kittens crosses her path, and that kitten is black with probability p, what is the expected number of hours before my neighbor believes she is doomed?

Solution:

4. Up For Adoption

(20) (a) I can't afford to feed all of Mrs. Ford's kittens, so I'm putting 100 of them up for adoption. I've lined up the kittens in a row against a wall in a hallway of my house such that the door of the hallway is at the right end of the line of kittens. I've gathered 100 friends of mine to adopt the kittens, and lined them up at the door. My friends are strange too! Each friend i has already selected a favorite kitten - the kitten at position X_i from the door, where X_i is selected from 1 to 100 randomly (each number with probability 1/100). I let my friends into the hallway one at a time. They keep walking down the hallway until one of two things happens:

- They reach their favorite kitten. At that point, they sit down and begin to play with that kitten.
- Their path is blocked. Since the hallway is rather narrow, any person sitting and playing with a kitten will block anyone from going past him. Instead, the person who has to choose a kitten sits down and begin to play with the kitten immediately before the one whose new owner is blocking his path.

If the kitten that is closest to the door is being played with, the rest of my friends still outside will not be able to enter the hall at all, and will have to leave empty handed. What is the most likely (note: "most likely" is **not** the same as "expected value") number of kittens that will be adopted?

Solution:

5. Beware of Dog

(15) (a) 1000 of Mrs. Ford's kittens are all playing together in a dangerous area, and she needs all of them to follow her as she leads them away. Unfortunately, the best she can do is to tell 63 of them to follow her. However, if Mrs. Ford tells a kitten to follow her, all of that kitten's friends will follow her too, and every kitten has 100 friends among the 999 other kittens who are playing. Mrs. Ford cannot decide which 63 to tell, so she just picks them randomly (all possible choices of 63 kittens are equally likely). Let X be the random variable counting the number of kittens that don't end up following Mrs. Ford. Show that $E[X] < 1$. Note that Mrs. Ford's choice of the 63 kittens is random, but the friendships among the kittens are fixed although arbitrary. That is, your proof should hold no matter how the kittens have made friends, as long as each kitten has 100 friends. Explain why this means Mrs. Ford could have chosen the 63 kittens wisely such that all 1000 kittens would have followed her. Again, your proof should hold for arbitrary friendships, as long as each kitten has 100 friends.

Solution:

6. Even More Kittens

(15) (a) You have a way of lining up 15251 of Mrs. Ford's kittens in a row such that the kitten at position i would be black with probability $\frac{1}{2i+1}$ and gray with probability $\frac{2i}{2i+1}$. (The color of each kitten is independent of other kittens.) You get a dollar for every quadruple (four kittens in a row) of the same color. What is the expected number of dollars you get?

Solution:

7. Bonus: Kitten Parity

(10) (a) Consider the 15251 kittens from problem 6. You hit a jackpot if the number of black kittens among the 15251 kittens is even. What is the probability of you hitting a jackpot?

Solution:

8. ## Feedback

We work hard to make 15-251 an enjoyable and instructive class. We want to hear from you.

- Please write something you like about this class.

Solution:

- Please tell us how we can improve (please be concise and constructive).

Solution: