15-251 : Great Theoretical Ideas In Computer Science

Fall 2013

Assignment 2 (The Mesozoic Homework)

Due: Thursday, Sep. 12, 2013 11:59 PM

Name: ____________________

Andrew ID: ____________________

Question:	1	2	3	4	5	Total
Points:	20	30	40	10	10	110
Score:						

1. Even More Pancakes

(20) (a) **Stone Age Pancakes**

During the Triassauric period, the Dinosauria had not yet mastered the formula for multi-sized pancakes. They were only capable of producing two sizes of pancakes: small and large. A young raptor by the name of Filo, while working at the Pangaea House of Pancakes, decided he would fix the unstable stacks of small and large pancakes by sorting them. Just like us, he had a spatula that could flip any number of pancakes on the top of a stack. His goal was to get all the small pancakes on top of the large pancakes. He identified a lower bound and an upper bound for the minimum number of flips required by worst stack of size n. To his surprise, the lower bound was the same as the upper bound! Recreate Filo's work by finding a lower and an upper bound. You will only receive full credit if these bounds are equal.

Solution:

2. Destroying Dominos

Filo's best friend Sarah, a Tryceratopid, was playing with dominos. Unlike other Dinosauria her age, she was bored with lines of dominos. Instead, she had made a semi-infinite(!) two dimensional grid of dominos. Formally, she had one domino placed at every (x, y) where $x, y \in \mathbb{Z}$ and $x \geq 0$ and $y \geq 0$. While Sarah was not there, Filo came into the room and wanted to destroy the whole grid. However, he did not have much time, as Sarah could come back at any moment. Because of a wind from the southwest, a falling domino at (x, y) would tip over the domino at $(x+1, y+1)$, but not touch any other domino directly. Filo quickly figured out the minimal set of domino(s) he would need to tip over to destroy the whole grid.

(10) (a) Prove that tipping over all dominos in the first row as well as all dominos in the first column (all dominos with $x = 0$ or $y = 0$) is both necessary and sufficient.

Solution:

(10) (b) After Sarah saw her grid destroyed, she was very frustrated. To prevent this in the future, she built a larger grid: Now she had a domino at every (x, y) where $x, y \in \mathbb{Z}$, but she had to build it with special dominos such that a falling domino at (x, y) would now tip over four vertically and horizontally adjacent dominos, i.e., dominos at $(x+1, y)$, $(x-1, y)$, $(x, y+1)$, and $(x, y-1)$. What is the minimum number of dominos Filo has to tip over in this case? Prove it.

Solution:

(10) (c) Learning from the previous experiences, Sarah got a little smarter. She just turned all the dominos by $45°$, so that a falling domino at (x, y) would now tip over the four dominos at $(x \pm 1, y \pm 1)$, i.e., the dominos at $(x+1, y+1)$, $(x+1, y-1)$, $(x-1, y+1)$, and $(x-1, y-1)$. Will this really help her? What is the minimum number of dominos Filo has to tip over in this case? Prove it.

Solution:

3. Induction Inundation

(20) (a) **Jurassic Jumping**

When Filo told his father Horus how he applied induction to grids, his father was very proud. It reminded Horus of his years on the Galloping Gavanisaurs, a Jurassic Jumping team. The game of Jurassic Jumping involves jumping from one square to another in an $n \times n$ grid, traveling exactly two squares in one direction, and one square in a perpendicular direction (e.g. up 2 left 1). There is a goal at some point in the grid, and players race to this goal from different starting points. After Horus beat his long-time rival Seth, Seth complained that he had been given an impossible starting position, and that there was no way for him to reach the goal. Horus responded that this was absurd; Seth clearly had a way to reach the goal. It is your job to understand Horus's reasoning. Prove that for any n by n grid where $n \geq 4$, there is a way to travel from any square on the grid to any other square on the grid by Jurassic Jumping.

Solution:

(20) (b) **Carnivorous Cannibal Circle**

While Horus expounded on his days with the Galloping Gavanisaurs, Filo began to think of a riddle:

A group of n carnivores are sitting at arbitrary points on a circle one kilometer in circumference. Each carnivore has a certain amount of calories stored in fat. Once a carnivore a reaches another carnivore b, a can choose to eat b, and gain 100% of b's fat. There is enough fat among all n dinosaurs to fuel any single one of them for one kilometer. Prove that there is a carnivore that can choose to run around the circle clockwise, eating everyone as he goes, and run out of fat just as he reaches his starting point.

Solution:

4. Structural Induction

(10) (a) **Triassauric Tree Troubles**

Horus was still talking when Filo left his daydream. He has started to talk about how bad the modern teams were because... But Filo was lost again, this time staring at the trees. The trees back then grew strangely. Each tree had trunk coming from the ground, and became a large branch. Each branch either terminated at a single leaf or split apart into n branches. Come up with a simple description of the set of possible number of leaves such a tree could have. Prove this description correct.

Solution:

5. Bonus Question

(10) (a) **Tetragametic Tree Troubles**

For some trees, the situation was even stranger! Horus had cross pollinated an m-ary and an n-ary tree. Each branch could split m times, split n times, or end in a leaf. Such a cross pollination was only successful if $m-1$ and $n-1$ were coprime, i.e., their gcd was 1. Prove that for any $k > mn - 2m - 2n + 4$, there exists a tree with k leaves.

Solution: